MATH152, 525–530, 534–536 Spring 2013,
Sample problems for Test 1

1. Find the area of the region bounded by $y = e^x$, $y = e^{-x}$, $x = -2$, and $x = 1$.

2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 - 1$, $y = 0$, $x = 1$, $x = 2$ about the x-axis.

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 0$. $x = 1$, $x = 2$ about

   (a) the y-axis

   (b) $x = 4$

4. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.

5. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the half rope to the top of the building?

6. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

7. A tank in a shape of a sphere of radius 9 m is half full of water. Find the work W required to pump the water out of the spout, if the height of the spout is 3 m.

8. Find the average value of $f(x) = \sin^2 x \cos x$ on $[-\pi/2, \pi/4]$.

9. Evaluate the integral

   (a) $\displaystyle\int t^2 \cos(1 - t^3) dt$

   (b) $\displaystyle\int \frac{x^2}{\sqrt{1-x}} dx$

   (c) $\displaystyle\int x^3 \sqrt{x^2 + 5}\, dx$

   (d) $\displaystyle\int_0^1 x^2 e^{-x} dx$

   (e) $\displaystyle\int \frac{\sin^3 x}{\sec^4 x}\, dx$

   (f) $\displaystyle\int_0^{\pi/8} \sin^2(2x) \cos^3(2x)\, dx$

   (g) $\displaystyle\int \sin^2 x \cos^4 x\, dx$

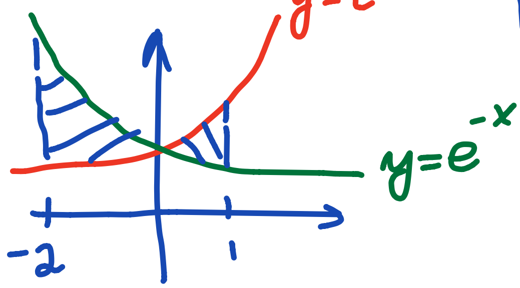
   (h) $\displaystyle\int_0^{\pi/4} \tan^4 x \sec^2 x\, dx$

   (i) $\displaystyle\int \tan x \sec^3 x\, dx$

   (j) $\displaystyle\int \sin 3x \cos x\, dx$

Solutions.

#1.



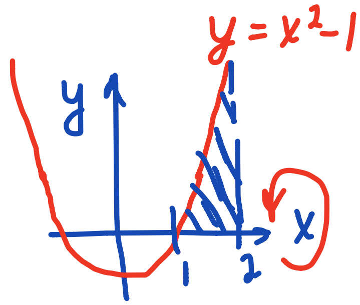
$$\text{Area} = \int_{-2}^{0} (e^{-x} - e^{x})\,dx + \int_{0}^{1} (e^{x} - e^{-x})\,dx$$

$$= \left(-e^{-x} - e^{x}\right)\Big|_{-2}^{0} + \left(e^{x} + e^{-x}\right)\Big|_{0}^{1}$$

$$= \left(-1-1+e^{2}+e^{-2}\right) + \left(e + e^{-1} - 1 - 1\right)$$

$$= \boxed{-4 + e^{2} + e^{-2} + e + e^{-1}}$$

#2.



We do disks:

$$V = \pi \int_{1}^{2} [\text{radius}]^{2}\,dx$$

$$[\text{radius}] = x^{2} - 1$$

$$V = \pi \int_{1}^{2} (x^{2}-1)^{2}\,dx = \pi \int_{1}^{2} (x^{4} - 2x^{2} + 1)\,dx$$

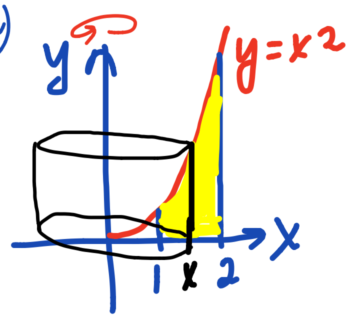
$$= \pi \left(\frac{x^{5}}{5} - \frac{2x^{3}}{3} + x \right)\Big|_{1}^{2} = \pi \left(\frac{32}{5} - \frac{16}{3} + 2 - \frac{1}{5} + \frac{2}{3} - 1 \right)$$

$$= \pi \left(\frac{31}{5} - \frac{14}{3} + 1 \right)$$

$$= \pi \, \frac{93 - 70 + 15}{15}$$

$$= \boxed{\frac{38\pi}{15}}$$

#3. a)



Shells:

$$V = 2\pi \int_{1}^{2} [\text{radius}][\text{height}]\,dx$$

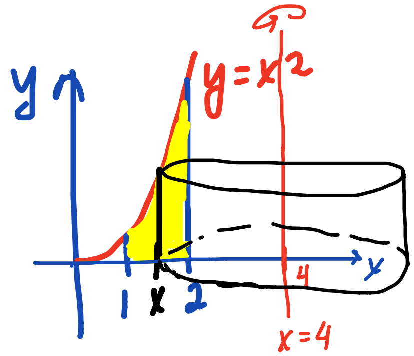
$$\text{radius} = x$$

$$\text{height} = x^{2}$$

$$V = 2\pi \int_{1}^{2} x(x^{2})\,dx = 2\pi \int_{1}^{2} x^{3}\,dx$$

$$= 2\pi \, \frac{x^{4}}{4} \Big]_{1}^{2} = \frac{\pi}{2}(16 - 1) = \boxed{\frac{15\pi}{2}}$$

#3b)



shells:

$V = 2\pi \int_1^2 (\text{radius})(\text{height})\,dx$

radius $= 4 - x$

height $= x^2$

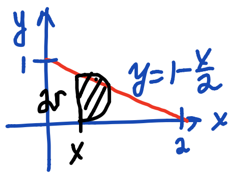
$V = 2\pi \int_1^2 (4-x)x^2\,dx = 2\pi \int_1^2 (4x^2 - x^3)\,dx$

$= 2\pi \left(\frac{4x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2 = 2\pi\left(\frac{4(8)}{3} - \frac{16}{4} - \frac{4}{3} + \frac{1}{4}\right)$

$= 2\pi\left(\frac{28}{3} - \frac{15}{4}\right) = 2\pi \frac{102-45}{12} = 2\pi \frac{67}{12}$

$= \boxed{\frac{67\pi}{6}}$

#4.



$0 \le x \le 2$

The equation of the line through $(2,0)$ and $(0,1)$ is $y = 1 - \frac{x}{2}$

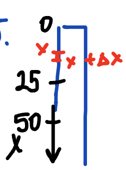
$A(s) = \frac{1}{2}\pi r^2$

$r = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{x}{2}\right) \Rightarrow r^2 = \frac{1}{4}\left(1 - \frac{x}{2}\right)^2$

$V = \frac{\pi}{4} \cdot \frac{1}{2}\int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = \frac{\pi}{8}\int_0^2 \left(1 - x + \frac{x^2}{4}\right) dx$

$= \frac{\pi}{8}\left(x - \frac{x^2}{2} + \frac{x^3}{12}\right)\Big|_0^2 = \frac{\pi}{8}\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \boxed{\frac{\pi}{12}}$

#5.



1) $0 \le x \le 25$

The portion of the rope weighs $0.5\Delta x$ (lb) and must be lifted x (ft)

Then the work done is

$W_1 = \int_0^{25} 0.5x\,dx = 0.5\,\frac{x^2}{2}\Big]_0^{25} = \frac{625}{4}$ (ft- lb)

2) $25 \le x \le 50$

The portion of the rope weighs $0.5\Delta x$ (lb) and must be lifted 25 (ft)

$$W_2 = \int_{25}^{50} 0.5(25)\,dx = 12.5x\Big|_{25}^{50} = 12.5(25) = 312.5\,(\text{ft-lb})$$

$$\text{Total work} = W_1 + W_2 = \frac{625}{4} + 312.5 = \boxed{468.75\,(\text{ft-lb})}$$

#6. $F(x) = kx$

20 cm $\mapsto$ 0

30 cm $\mapsto$ $30-20 = 10\,(cm) = .1\,(m)$

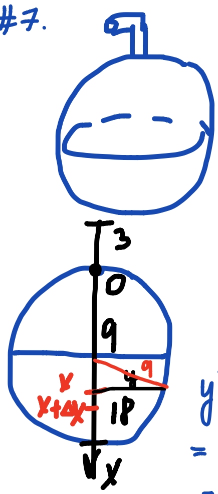
25 cm $\mapsto$ $25-20 = 5\,(cm) = .05\,(m)$

$$25 = k(0.1) \to k = \frac{25}{0.1} = 250$$

$$F(x) = 250x$$

$$W = \int_0^{0.05} 250x\,dx = \frac{250}{2}x^2\Big]_0^{0.05} = 125\,(0.0025) = 0.3125\,(J)$$

#7.



$r = 9m$

$h = 3m$

$$y^2 = 9^2 - (9-x)^2$$
$$= 81 - (81 - 18x + x^2)$$
$$= 18x - x^2$$

$9 \le x \le 18$

a slice of water between x and $x+\Delta x$ is a cylinder of height Δx and radius y.

$$y = \sqrt{18x - x^2}$$

The volume of the slice

$$V = \pi y^2 = \pi(18x - x^2)$$

Weight of the slice

$$= V\rho g = \pi(18x - x^2)\,(10^3)(9.8)$$

---

distance traveled by the slice $= x+3$

Then the work is

$$W = \pi(10^3)(9.8)\int_9^{18}(18x - x^2)(3+x)\,dx$$

$$= \pi(10^3)(9.8)\int_9^{18}(54x + 18x^2 - 3x^2 - x^3)\,dx$$

$$= \pi(10^3)(9.8)\int_9^{18}(54x + 15x^2 - x^3)\,dx$$

$$= \pi(10^3)(9.8)\left[54\,\frac{x^2}{2} + \frac{15x^3}{3} - \frac{x^4}{4}\right]_9^{18}$$

$$\boxed{= \pi(10^3)(9.8)\left[27(324)+5(5832)-26{,}244-27(81)-5(729)+\frac{6561}{4}\right]}$$

#8. $f_{ave} = \dfrac{1}{\frac{\pi}{4}+\frac{\pi}{2}} \displaystyle\int_{-\pi/2}^{\pi/4} \sin^2 x \cos x \, dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x\, dx \\ -\frac{\pi}{2} \to \sin\left(-\frac{\pi}{2}\right) = -1 \\ \frac{\pi}{4} \to \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \end{array} \right| = \dfrac{4}{3\pi}\displaystyle\int_{-1}^{\frac{\sqrt{2}}{2}} u^2\, du$

$$= \frac{4}{3\pi}\,\frac{u^3}{3}\Big]_{-1}^{\sqrt{2}/2} = \frac{4}{9\pi}\left(\left(\frac{\sqrt{2}}{2}\right)^3 - (-1)^3\right) = \frac{4}{9\pi}\left(\frac{2\sqrt{2}}{8}+1\right) = \boxed{\frac{4}{9\pi}\left(\frac{\sqrt{2}}{4}+1\right)}$$

#9.

(a) $-\frac{1}{3}\displaystyle\int -3t^2\cos(1-t^3)\,dt = \left| \begin{array}{l} u = 1-t^3 \\ du = -3t^2 dt \end{array} \right| = -\frac{1}{3}\displaystyle\int \cos u\, du = -\frac{1}{3}\sin u + C$

$$= \boxed{-\frac{1}{3}\sin(1-t^3)+C}$$

(b) $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\,dx = \left| \begin{array}{l} u = 1-x \\ x = 1-u \\ du = -dx \end{array} \right| = -\int \frac{(1-u)^2}{\sqrt{u}}\,du = -\int (1-2u+u^2)\,u^{-1/2}\,du$

$$= -\int\left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right)du = -\frac{u^{1/2}}{1/2} - 2\,\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + c$$

$$= -2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c = \boxed{-2(1-x)^{1/2} - \frac{4}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C}$$

(c) $\displaystyle\int x^2\sqrt{x^2+5}\,dx = \left| \begin{array}{l} u = x^2+5 \\ du = 2x\,dx \\ x^2 = u-5 \end{array} \right| = \frac{1}{2}\int 2(x^2)(x)\sqrt{x^2+5}\,dx = \frac{1}{2}\int (u-5)\sqrt{u}\,du$

$$= \frac{1}{2}\int\left(u^{3/2} - 5u^{1/2}\right)du = \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - 5\,\frac{u^{3/2}}{3/2}\right) + c = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{5}u^{5/2} - 5\,\frac{\cancel{2}}{3}u^{3/2}\right) + C$$

$$= \frac{1}{5}u^{5/2} - \frac{5}{3}u^{3/2} + c = \boxed{\frac{1}{5}(x^2+5)^{5/2} - \frac{5}{3}(x^2+5)^{3/2} + C}$$

(d) $\int_0^1 x^2 e^{-x}\,dx$ by parts $\left| \begin{array}{ll} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{array} \right|$

$= x^2(-e^{-x})\Big]_0^1 + \int_0^1 2x(+e^{-x})\,dx = -e^{-1} + 2\int_0^1 x e^{-x}\,dx$ $\left| \begin{array}{ll} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{array} \right|$

$= -e^{-1} + 2\left(x(-e^{-x})\Big]_0^1 + \int_0^1 (+e^{-x})\,dx \right) = -e^{-1} - 2e^{-1} - 2e^{-x}\Big]_0^1$

$= -e^{-1} - 2e^{-1} - 2e^{-1} + 2 = \boxed{2 - 5e^{-1}}$

(e) $\int \frac{\sin^3 x}{\sec^4 x}\,dx = \int \sin^3 x \cos^4 x\,dx = \int \sin x \underbrace{\sin^2 x}_{1-\cos^2 x} \cos^4 x\,dx = \int \sin x (1-\cos^2 x)\cos^4 x\,dx$

$\left| \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array} \right| = -\int (1-u^2)u^4\,du = -\int (u^4 - u^6)\,du = -\frac{u^5}{5} + \frac{u^7}{7} + C$

$= \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$

(f) $\int_0^{\pi/8} \sin^2(2x)\cos^3(2x)\,dx = \int_0^{\pi/8} \cos(2x)\sin^2(2x)\underbrace{\cos^2(2x)}_{1-\sin^2(2x)}\,dx$

$= \frac{1}{2}\int_0^{\pi/8} 2\cos(2x)(1-\sin^2(2x))\sin^2(2x)\,dx =$ $\left| \begin{array}{l} u = \sin(2x) \\ du = 2\cos(2x)\,dx \\ 0 \to \sin 0 = 0 \\ \frac{\pi}{8} \to \sin\frac{2\pi}{8} = \frac{\sqrt{2}}{2} \end{array} \right|$

$= \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} (1-u^2)u^2\,du = \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} (u^2 - u^4)\,du = \frac{1}{2}\left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^{\frac{\sqrt{2}}{2}}$

$= \frac{1}{2}\left[\left(\frac{\sqrt{2}}{2}\right)^3 \frac{1}{3} - \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5 \right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}\sqrt{2}}{(8)(3)} - \frac{\cancel{4}\sqrt{2}}{32(5)} \right]$

$= \boxed{\frac{\sqrt{2}}{24} - \frac{\sqrt{2}}{80}}$

(f) $\int \sin^2 x \cos^4 x\,dx = \int \underbrace{(\sin^2 x \cos^2 x)}_{(\frac{1}{2}\sin 2x)^2} \underbrace{(\cos^2 x)}_{\frac{1}{2}(1+\cos 2x)}\,dx = \frac{1}{8}\int \sin^2(2x)(1+\cos(2x))\,dx$

$$= \frac{1}{8}\int \underbrace{\sin^2(2x)}_{\frac{1}{2}(1-\cos(4x))}dx + \frac{1}{8}\cdot\frac{1}{2}\int 2\sin^2(2x)\cos(2x)\,dx \quad \left| \begin{array}{l} u = \sin(2x) \\ du = 2\cos 2x\,dx \end{array} \right|$$

$$= \frac{1}{16}\int (1-\cos(4x))\,dx + \frac{1}{16}\int u^2\,du = \frac{1}{16}\left(x - \frac{1}{4}\sin(4x)\right) + \frac{1}{16}\,\frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{16}\left(x - \frac{1}{4}\sin(4x)\right) + \frac{1}{48}\sin^3(2x) + C}$$

(h) $\int_0^{\pi/4} \tan^4 x \sec^2 x\,dx = \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \\ 0 \to \tan 0 = 0 \\ \frac{\pi}{4} \to \tan\frac{\pi}{4} = 1 \end{array} \right| = \int_0^1 u^4\,du = \left.\frac{u^5}{5}\right]_0^1 = \boxed{\frac{1}{5}}$

(i) $\int \tan x \sec^3 x\,dx = \int (\tan x \sec x)\sec^2 x\,dx \quad \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x\,dx \end{array} \right| = \int u^2\,du$

$$= \frac{u^3}{3} + C = \boxed{\frac{\sec^3 x}{3} + C}$$

(j) $\int \sin 3x \cos x\,dx = \frac{1}{2}\int [\sin(3x-x) + \sin(3x+x)]\,dx = \frac{1}{2}\int (\sin 2x + \sin 4x)\,dx$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C = \boxed{-\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C}$$