6. Find the power series representation for the function $f(x) = \ln(1-2x)$ centered at 0.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (+1)^n (2x)^n = \sum_{n=0}^{\infty} (+1)^n 2^n x^n$$

$$\ln(1-2x) = -2\int \frac{1}{1-2x}\,dx = -2\int\left(\sum_{n=0}^{\infty} (+1)^n 2^n x^n\right)dx$$

$$= -2\sum_{n=0}^{\infty} (+1)^n 2^n \left(\int x^n\,dx\right) = \sum_{n=0}^{\infty} -1(+2)^{n+1}\frac{x^{n+1}}{n+1} + C$$

c: plug $x=0$;

$\ln 1 = C \Rightarrow c = 0.$

$$\boxed{\ln(1-2x) = \sum_{n=0}^{\infty} (-1)(+2)^{n+1}\frac{x^{n+1}}{n+1}}$$

7. Find the Taylor series for $f(x) = xe^{2x}$ at $x = 2$.

$f(x) = xe^{2x}$

$f'(x) = e^{2x} + 2xe^{2x} = 1\cdot 2^0 e^{2x} + 2^1 xe^{2x}$

$f''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x}$

$= 4e^{2x} + 4xe^{2x} = 2\cdot 2^1 e^{2x} + 2^2 xe^{2x}$

$f'''(x) = 8e^{2x} + 4e^{2x} + 8xe^{2x}$

$= 12e^{2x} + 8xe^{2x} = 3\cdot 2^2 e^{2x} + 2^3 xe^{2x}$

$f^{(n)}(x) = n\cdot 2^{n-1} e^{2x} + 2^n x e^{2x}$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(n2^{n-1} + 2^n\cdot 2)e^4}{n!}(x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n2^{n-1} + 2^{n+1})e^4}{n!}(x-2)^n$$

8. Find the Maclaurin series for $f(x) = x\sin(x^3)$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

$$x\sin x^3 = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}$$

9. Find the sum of the series

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^2}{n!}$ $\displaystyle = x^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} = x^2\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - 1 + 1\right]$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = x^2 e^{-1}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$ $\displaystyle = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

10. Evaluate the indefinite integral as a power series $\int e^{x^2}\,dx$.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int e^{x^2}dx = \int\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right)dx = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int x^{2n}dx\right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\,n!} + C$$

11. Approximate $f(x) = \sin x$ by a Taylor polynomial of degree 3 at $\pi/4$. How accurate is this approximation if $0 \le x \le \pi/2$?

$$\sin x \approx f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + f''\left(\frac{\pi}{4}\right)\cdot\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 + f'''\left(\frac{\pi}{4}\right)\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3$$

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{4}\right) = f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$

$$\sin x \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$

Taylor's Inequality $\quad |R_3| \le \frac{M}{4!}\left|x-\frac{\pi}{4}\right|^4, \ |f^{(4)}(x)| \le M$

$f^{(4)}(x) = \sin x. \quad |\sin x| \le 1 \Rightarrow M = 1$

$0 \le x \le \frac{\pi}{2} \longrightarrow -\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{\pi}{4} \longrightarrow \left|x-\frac{\pi}{4}\right| \le \frac{\pi}{4}.$

$$\boxed{|R_3| \le \frac{1}{4!}\left(\frac{\pi}{4}\right)^4}$$

12. Find radius and center of sphere given by the equation $x^2 + y^2 + z^2 = 6x + 4y + 10z$

Complete squares:

$x^2 - 6x + y^2 - 4y + z^2 - 10z = 0$

$(x-3)^2 + (y-2)^2 + (z-5)^2 - 9 - 4 - 25 = 0$

$(x-3)^2 + (y-2)^2 + (z-5)^2 - 38 = 0$

$(x-3)^2 + (y-2)^2 + (z-5)^2 = 38$

Radius $= \sqrt{38}$

center $(3, 2, 5)$.

13. Find the angle between the vectors $\vec{a} = \vec{\imath} + \vec{\jmath} + 2\vec{k}$ and $\vec{b} = 2\vec{\jmath} - 3\vec{k}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1)(0)+1(2)+2(-3)}{\sqrt{1+1+4}\sqrt{4+9}} = -\frac{4}{\sqrt{6}\sqrt{13}} = -\frac{4}{\sqrt{78}}$$

$$\theta = \cos^{-1}\left(-\frac{4}{\sqrt{78}}\right)$$

14. Find the directional cosines for the vector $\vec{a} = -2\vec{\imath} + 3\vec{\jmath} + \vec{k}$. , $|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$

$$\cos\alpha = \frac{-2}{\sqrt{14}}$$

$$\cos\beta = \frac{3}{\sqrt{14}}$$

$$\cos\gamma = \frac{1}{\sqrt{14}}$$

15. Find the scalar and the vector projections of the vector $\vec{a} = \langle 2, -3, 1 \rangle$ onto the vector $\langle 1, 6, -2 \rangle = \vec{b}$

$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2-18-2}{\sqrt{1+36+4}} = -\frac{18}{\sqrt{41}}$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = -\frac{18}{41}\langle 1, 6, -2 \rangle$$