Chapter 10. **Infinite sequences and series**
Section 10.1 **Sequences**

A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, ..., a_n, ...$$

For each $n = 1, 2, 3, ..., a_n = f(n)$.

The sequence $a_1, a_2, ..., a_n, ...$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 1. Find the formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

1. $\left\{1, \frac{1}{2}, \frac{1}{8}, \cdots\right\}$

2. $\left\{\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \cdots\right\}$

Example 2. List the first three terms of the sequence $\left\{\frac{2n+1}{4^{n-1}}\right\}$.

Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** or is **convergent**. Otherwise, we say the sequence **diverges** or is **divergent**

Limit Laws. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} [a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{x\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} c = c$

The Squeeze Theorem. If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 3. Find the limit

1. $\lim_{n\to\infty} (-1)^n \dfrac{n^2}{1+n^3}$

2. $\lim_{n\to\infty} \dfrac{\cos^2 n}{2^n}$

3. $\lim_{n\to\infty} \frac{\pi^n}{3^n}$

Definition. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Example 4. Determine whether the sequence is increasing, decreasing, or not monotonic.

1. $a_n = \frac{1}{3n+5}$

2. $a_n = 3 + \frac{(-1)^n}{n}$

3. $a_n = \dfrac{n-2}{n+2}$

Definition. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$a_n \geq m \qquad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

Monotonic Sequence Theorem. Every bounded, monotonic sequence is convergent.

Example 5. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

is decreasing and bounded. Find the limit of $\{a_n\}$.