Chapter 10. **Infinite sequences and series**
Section 10.1 **Sequences**

A sequence is a list of numbers written in a definite order:

$$a_1, a_2, ..., a_n, ...$$

For each $n = 1, 2, 3, ...,$ $\boxed{a_n = f(n)}$.

The sequence $a_1, a_2, ..., a_n, ...$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Examples of sequences:

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..., \frac{n}{n+1}, ...$

$\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, ..., \frac{2n-1}{2^n}, ...$

Example 1. Find the formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

1. $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\}$

$a_1 = 1 = \frac{1}{2^0}$

$a_2 = \frac{1}{2} = \frac{1}{2^1}$

$a_3 = \frac{1}{4} = \frac{1}{2^2}$

$a_4 = \frac{1}{8} = \frac{1}{2^3}$

$\boxed{a_n = \frac{1}{2^{n-1}}, \quad n = 1, 2, 3, ...}$

$\boxed{a_n = \frac{1}{2^n}, \quad n = 0, 1, 2, ...}$

2. $\left\{\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \cdots\right\}$

$a_{\boxed{1}} = \frac{2}{3} = \frac{2 \boxed{1}}{3^{\boxed{1}}}$

$a_{\boxed{2}} = \frac{4}{9} = \frac{2 \boxed{2}}{3^{\boxed{2}}}$

$a_{\boxed{3}} = \frac{6}{27} = \frac{2 \boxed{3}}{3^{\boxed{3}}}$

$\boxed{a_n = \frac{2 \cdot n}{3^n}, \quad n = 1, 2, 3, ...}$

Example 2. List the first three terms of the sequence $\left\{\frac{2n+1}{4^{n-1}}\right\}$.

$a_n = \frac{2n+1}{4^{n-1}}, \quad n = 1, 2, 3, ...$

$a_1 = \frac{2(1)+1}{4^{1-1}} = 3$

$a_2 = \frac{2(2)+1}{4^{2-1}} = \frac{5}{4}$

$a_3 = \frac{2(3)+1}{4^{3-1}} = \frac{7}{16}$

Definition. A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** or is convergent. Otherwise, we say the sequence **diverges** or is divergent.

Limit Laws. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} [a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{x\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} c = c$

The Squeeze Theorem. If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} r^n = \begin{cases} \infty, & \text{if } r>1 \\ 0, & \text{if } r<1 \end{cases}, \quad r \text{ is a real number}$$

Example 3. Find the limit

$$\lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \dots + a_k n^k}{b_0 + b_1 n + b_2 n^2 + \dots + b_m n^m} = \begin{cases} 0, & \text{if } m > k \\ \infty, & \text{if } k > m \\ \frac{a_k}{b_m}, & \text{if } k = m \end{cases}$$

1. $\lim_{n\to\infty} (-1)^n \frac{n^2}{1+n^3} = \boxed{0}$

$a_n = (-1)^n \frac{n^2}{1+n^3}$

$|a_n| = \left|(-1)^n \frac{n^2}{1+n^3}\right| = \frac{n^2}{1+n^3} \quad (n>0).$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{1+n^3} = \lim_{n\to\infty} \frac{n^2}{n^3(\cancel{\frac{1}{n^3}}^{0}+1)} = \lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} |a_n| = 0.$

By the Theorem, $\lim_{n\to\infty} a_n = 0$

2. $\lim_{n\to\infty} \frac{\cos^2 n}{2^n}$

use the Squeeze Theorem.

$\frac{0}{2^n} \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$

$0 \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$

$\lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0 \quad \left(\frac{1}{2} < 1\right)$

$$\boxed{\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0}$$

By the Squeeze Theorem.

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3. $\lim_{n\to\infty} \frac{\pi^n}{3^n} = \lim_{n\to\infty} \left(\frac{\pi}{3}\right)^n = \boxed{\infty}$ $\qquad \frac{\pi}{3} > 1$

$\boxed{a^2 - b^2 = (a-b)(a+b)}$

4. $\lim_{n\to\infty} \left(\sqrt{n^2-2n+2} - \sqrt{n^2+1}\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2-2n+2} - \sqrt{n^2+1}\right)\left(\sqrt{n^2-2n+2} + \sqrt{n^2+1}\right)}{\sqrt{n^2-2n+2} + \sqrt{n^2+1}}$

$= \lim_{n\to\infty} \frac{\left(\sqrt{n^2-2n+2}\right)^2 - \left(\sqrt{n^2+1}\right)^2}{\sqrt{n^2-2n+2} + \sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\cancel{n^2} - 2n + 2 - \cancel{n^2} - 1}{\sqrt{n^2-2n+2} + \sqrt{n^2+1}}$

$= \lim_{n\to\infty} \frac{-2n+1}{\sqrt{n^2-2n+2} + \sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n(-2+\cancel{\frac{1}{n}}^{0})}{\sqrt{n^2(1-\cancel{\frac{2}{n}}^{0}+\cancel{\frac{2}{n^2}}^{0})} + \sqrt{n^2(1+\cancel{\frac{1}{n^2}}^{0})}}$

$= \lim_{n\to\infty} \frac{-2n}{\sqrt{n^2} + \sqrt{n^2}} = \lim_{n\to\infty} \frac{-2n}{2n} = \boxed{-1}$

Definition. A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

monotonic = increasing or decreasing.

Example 4. Determine whether the sequence is increasing, decreasing, or not monotonic.

neither increasing nor decreasing.

1. $a_n = \dfrac{1}{3n+5}$

$a_1 = \frac{1}{3+5} = \frac{1}{8}$

$a_2 = \frac{1}{3(2)+5} = \frac{1}{11}$ $\qquad a_1 > a_2 > a_3$

$a_3 = \frac{1}{3(3)+5} = \frac{1}{14}$

Prove that the sequence is decreasing for the general n.

Compare $\frac{1}{3n+5} = a_n$ $\qquad a_{n+1} = \frac{1}{3(n+1)+5}$

$5 + 3n < 3(n+1) + 5$

$\frac{1}{3n+5} > \frac{1}{3(n+1)+5}$

For general n, $a_n > a_{n+1}$ $\qquad \boxed{\text{decreasing}}$

2. $a_n = 3 + \dfrac{(-1)^n}{n}$

$a_1 = 3 + \frac{(-1)^1}{1} = 3 - 1 = 2$ $\qquad a_1 < a_2 > a_3$

$a_2 = 3 + \frac{(-1)^2}{2} = 3 + \frac{1}{2} = \frac{7}{2}$ $\qquad \boxed{\text{not monotonic}}$

$a_3 = 3 + \frac{(-1)^3}{3} = 3 - \frac{1}{3} = \frac{8}{3}$

3. $a_n = \dfrac{n-2}{n+2}$ – improper fraction.

$$\begin{array}{r|l} & 1 \text{ whole part} \\ n+2 & n-2 \\ & -(n+2) \\ \hline & -4 \text{ remainder} \end{array}$$

$a_n = \frac{n-2}{n+2} = 1 - \frac{4}{n+2}$

compare with $a_{n+1} = 1 - \frac{4}{(n+1)+2} = 1 - \frac{4}{n+3}$

$n+2 < n+3$

$\frac{4}{n+2} > \frac{4}{n+3}$

$\underbrace{1 - \frac{4}{n+2}}_{a_n} < \underbrace{1 - \frac{4}{n+3}}_{a_{n+1}}$

For the general n, $a_n < a_{n+1}$. $\qquad \boxed{\text{increasing}}$

Definition. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$\boxed{a_n \le M} \quad \text{for all } n \ge 1$$

It is bounded below if there is a number m such that

$$\boxed{a_n \ge m} \quad \text{for all } n \ge 1$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence

Monotonic Sequence Theorem. Every bounded, monotonic sequence is convergent.

Example 5. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

is decreasing and bounded. Find the limit of $\{a_n\}$.

1. Decreasing. ✓

$a_1 = 2$

$a_2 = \frac{1}{3-a_1} = \frac{1}{3-2} = 1$

$a_3 = \frac{1}{3-a_2} = \frac{1}{3-1} = \frac{1}{2}$

$a_1 > a_2 > a_3$.

2. Bounded. ✓

$2 = a_1 > a_2 > a_3 > \dots > a_n > \dots$

$0 < a_n \le 2$

3. $\lim_{n\to\infty} a_n$.

If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_{n+1} = L$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3-a_{n-1}} = \frac{1}{\lim_{n\to\infty}(3-a_{n-1})}$$

$$= \frac{1}{3 - \lim_{n\to\infty} a_{n-1}}$$

Let $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n-1} = L$.

$L = \frac{1}{3-L}$

$L(3-L) = 1$

$3L - L^2 = 1$

$L^2 - 3L + 1 = 0$

$L_1 > 2$

$L_1 = \frac{3+\sqrt{9-4}}{2} = \cancel{\frac{3+\sqrt{5}}{2}}, \quad L_2 = \boxed{\frac{3-\sqrt{5}}{2}}$

Since $0 < a_n \le 2$, then $0 < L \le 2$.