Section 10.2 **Series**

An expression of the form

$$a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n$$

is called an **infinite series** or **series**.

Consider partial sums:

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n = a_1 + a_2 + ... + a_n$$

Definition. Given a series

$$\sum_{n=0}^{\infty} a_n$$

and let

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n \text{ (the } n\text{th partial sum)}$$

If the sequence

$$\{S_n\}_{n=1}^{\infty}$$

converges and

$$\lim_{n\to\infty} S_n = S$$

then the series is called **convergent** and we write

$$\sum_{n=0}^{\infty} a_n = S$$

The number S is called the **sum of the series**. Otherwise, the series is called **divergent**.

The **geometric series**

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1 \\ \infty, & \text{if } |r| \geq 1 \end{cases}$$

Example 1. Find four partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n - 2}$.

Example 2. If the nth partial sum of the series $\sum_{n=0}^{\infty} a_n$ is $s_n = \dfrac{n-1}{n+1}$, find a_n and the sum of the series S.

Example 3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

1. $4 + \dfrac{8}{5} + \dfrac{16}{25} + \dfrac{32}{125} + ...$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n}$

3. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Example 4. Write the number $0.\overline{307}$ as a ratio of integers.

- The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
- The p**-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$

Theorem. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n = 0$, we can not conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

Test for divergence. If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5. Let $a_n = \dfrac{2n}{3n+1}$.

1. Determine whether $\{a_n\}$ is convergent.

2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

Theorem. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant), $\sum_{n=1}^{\infty}(a_n + b_n)$, $\sum_{n=1}^{\infty}(a_n - b_n)$, and:

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

NOTE. A finite number of terms can not affect the convergence of the series.

Example 6. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{6^n}$.