Section 10.2 Series

An expression of the form

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \quad , \quad a_n = f(n) \text{ for all } n.$$

is called an **infinite series** or series.

Consider partial sums:

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n = a_1 + a_2 + \ldots + a_n$$

Definition. Given a series

$$\sum_{n=0}^{\infty} a_n$$

and let

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n \text{ (the } n\text{th partial sum)}$$

If the sequence

$$\{S_n\}_{n=1}^{\infty} \quad \text{sequence of partial sums}$$

converges and

$$\lim_{n\to\infty} S_n = S$$

then the series is called convergent and we write

$$\boxed{\sum_{n=0}^{\infty} a_n = S = \lim_{n\to\infty} S_n}$$

The number S is called the sum of the series. Otherwise, the series is called divergent.

sequence of partial sums is divergent

The geometric series

$$\boxed{\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1 \\ \infty, & \text{if } |r| \geq 1 \end{cases}}$$

Example 1. Find four partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n - 2}$.

$$S_1 = a_1 = \frac{1}{1^2 + 2(1) - 2} = 1$$

$$S_2 = \underbrace{a_1}_{S_1} + a_2 = S_1 + a_2 = 1 + \frac{1}{2^2 + 2(2) - 2} = 1 + \frac{1}{6} = \frac{7}{6}$$

$$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3 = \frac{7}{6} + \frac{1}{3^2 + 3(2) - 2} = \frac{7}{6} + \frac{1}{13}$$

$$S_4 = \underbrace{a_1 + a_2 + a_3}_{S_3} + a_4 = S_3 + a_4 = \frac{7}{6} + \frac{1}{13} + \frac{1}{4^2 + 2(4) - 2} = \frac{7}{6} + \frac{1}{13} + \frac{1}{22}$$

Example 2. If the nth partial sum of the series $\sum_{n=\text{[redacted]}}^{\infty} a_n$ is $s_n = \dfrac{n-1}{n+1}$, find a_n and the sum of the series S.

$$S_n = \frac{n-1}{n+1}, \quad n = 1, 2, 3, \ldots$$

The sum of the series $S = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{n-1}{n+1} = 1$

$$S_n = \underbrace{a_1 + a_2 + a_3 + \ldots + a_{n-1}}_{S_{n-1}} + a_n$$

$$S_n = S_{n-1} + a_n \Rightarrow a_n = S_n - S_{n-1}$$

$$S_n = \frac{n-1}{n+1}, \quad S_{n-1} = \frac{(n-1)-1}{(n+1)-1} = \frac{n-2}{n}$$

$$a_n = S_n - S_{n-1} = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{n(n-1)-(n+1)(n-2)}{n(n+1)}$$

$$= \frac{n^2 - n - (n^2 - 2n + n - 2)}{n(n+1)} = \frac{\cancel{n^2} - \cancel{n} - \cancel{n^2} + \cancel{2n} - \cancel{n} + 2}{n(n+1)}$$

$$\boxed{a_n = \frac{2}{n(n+1)}}$$

Example 3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

1. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \ldots = 4\left(1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \ldots\right)$

$$= 4\left(\left(\frac{2}{5}\right)^0 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \ldots\right)$$

geometric series, $a = 4$, $r = \frac{2}{5} < 1$ (convergent)

$$= \frac{4}{1 - 2/5} = \frac{4}{3/5} = \boxed{\frac{20}{3}}$$

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \text{ if } |r|<1}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n} = \sum_{n=1}^{\infty} 4\,\frac{4^n}{5^n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty} 4\,\frac{4}{5}\left(\frac{4}{5}\right)^{n-1}$

geometric series, $a = \frac{16}{5}$, $r = \frac{4}{5} < 1$

$$S = \frac{\frac{16}{5}}{1 - \frac{4}{5}} = \frac{\frac{16}{5}}{1/5} = \boxed{16}$$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Write down the equation of the n-th partial sum.

1. $\frac{1}{n(n+2)}$ partial fractions.

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$\frac{1}{n(n+2)} = \frac{A(n+2)+Bn}{n(n+2)}$$

$1 = A(n+2) + Bn$

$n=-2$: $1 = -2B \Rightarrow \boxed{B = -\frac{1}{2}}$

$n=0$: $1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

$$\boxed{a_n = \frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)}$$

2. Partial sums.

$S_1 = a_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) = \frac{1}{2}\left(1 - \frac{1}{3}\right)$

$S_2 = a_1 + a_2 = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3 = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}\right)$

$S_4 = S_3 + a_4 = \frac{1}{2}\left(1 + \frac{1}{2} - \cancel{\frac{1}{4}} - \frac{1}{5}\right) + \frac{1}{2}\left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right)$

$$\boxed{S_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)}$$

3. $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \cancel{\frac{1}{n+1}}^{\,0} - \cancel{\frac{1}{n+2}}^{\,0}\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \boxed{\frac{3}{4}}$

Example 4. Write the number $0.\overline{307}$ as a ratio of integers.

$$0.\overline{307} = 0.\underbrace{307}\underbrace{307}\underbrace{307}307\ldots$$

$$= \frac{307}{1000} + \frac{307}{1000000} + \frac{307}{10^9} + \ldots$$

$$= \frac{307}{1000}\left(1 + \frac{1}{1000} + \frac{1}{1000000} + \ldots\right)$$

$$= \frac{307}{1000}\left(\left(\frac{1}{1000}\right)^0 + \frac{1}{1000} + \left(\frac{1}{1000}\right)^2 + \ldots\right)$$

geometric sum, $a = \frac{307}{1000}$, $r = \frac{1}{1000}$

$$0.\overline{307} = \frac{\frac{307}{1000}}{1 - \frac{1}{1000}} = \boxed{\frac{307}{999}}$$

- $\boxed{\text{The harmonic series } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}}$
- $\boxed{\text{The } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent for } p > 1 \text{ and divergent for } p \leq 1}$

Theorem. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n = 0$, we can not conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

Test for divergence. If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5. Let $a_n = \dfrac{2n}{3n+1}$.

1. Determine whether $\overset{\text{sequence}}{\{a_n\}}$ is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$$

The sequence $\{a_n\}$ is convergent.

2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$$

$\boxed{\text{divergent by the Test for divergence}}$

Theorem. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant), $\sum_{n=1}^{\infty}(a_n + b_n)$, $\sum_{n=1}^{\infty}(a_n - b_n)$, and:

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty} a_n$ converges, then so is the series $\sum_{n=5}^{\infty} a_n$

NOTE. A finite number of terms can not affect the convergence of the series.

Example 6. Find the sum of the series $\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{6^n}$.

$$= \sum_{n=1}^{\infty}\left(\frac{3^n}{6^n} + \frac{(-2)^n}{6^n}\right) = \sum_{n=1}^{\infty}\left(\left(\frac{3}{6}\right)^n + \left(-\frac{2}{6}\right)^n\right)$$

$$= \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n\right)$$

$\frac{1}{2} < 1$, so $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent

$\left|-\frac{1}{3}\right| = \frac{1}{3} < 1$, so $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n$ is convergent

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{n-1}$$

$$= \frac{1/2}{1-1/2} + \frac{-1/3}{1-(-1/3)} = 1 + \frac{-1/3}{4/3} = 1 - \frac{1}{4}$$

$$= \boxed{\frac{3}{4}}$$