The Integral Test Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

n does not have to be 1, it could be a different number. Function $f(x)$ has to be ultimately decreasing function, that is, decreasing for $x > N$.

Example 1. Determine whether the series is convergent or divergent.

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$a_n = \frac{1}{n \ln n} = f(n)$

$f(x) = \frac{1}{x \ln x}$

1. Prove that $f(x) = \frac{1}{x \ln x}$ is positive, continuous, decreasing on $[2, \infty)$

- $\frac{1}{x \ln x}$ on $[2, \infty)$ $\quad x \geq 2$ - positive, $\ln x > 0$ on $[2, \infty)$

$\frac{1}{x \ln x} > 0$ on $[2, \infty)$

- $\frac{1}{x \ln x}$ discontinuous @ $\boxed{x=0}$ and $\ln x = 0$, $\boxed{x=1}$ not in $[2, \infty)$

$\frac{1}{x \ln x}$ is continuous on $[2, \infty)$

- $f(x) = \frac{1}{x \ln x} = (x \ln x)^{-1}$

$f'(x) = -(x \ln x)^{-2} (x \ln x)'$

$= -(x \ln x)^{-2} \left(\ln x + x \frac{1}{x}\right)$

$= -\frac{\ln x + 1}{(x \ln x)^2} < 0$ on $[2, \infty)$

$\frac{1}{x \ln x}$ decreases on $[2, \infty)$

2. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ compare with $\int_2^{\infty} \frac{1}{x \ln x} dx$

$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \ln x} dx$

$u = \ln x$, $du = \frac{1}{x} dx$, $2 \to \ln 2$, $t \to \ln t$

$\boxed{\lim_{t \to \infty} \ln t = \infty \qquad \lim_{t \to \infty} \ln |\ln t| = \infty}$

$= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u} du = \lim_{t \to \infty} \ln|u| \Big]_{\ln 2}^{\ln t} = \lim_{t \to \infty} \left(\ln|\ln t| - \ln|\ln 2|\right) = \infty$

$\boxed{\int_2^{\infty} \frac{1}{x \ln x} dx \text{ diverges}}$

$\boxed{\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ is divergent}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

$$\underline{a_n = \frac{1}{n^2+4} = f(n)}$$

$$f(x) = \frac{1}{x^2+4}$$

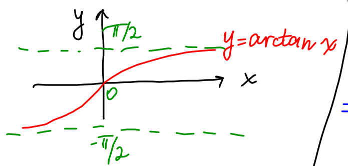
1. Show that $f(x) = \frac{1}{x^2+4}$ is positive, continuous, decreasing on $[1, \infty)$

- $\frac{1}{x^2+4} > 0$ for all $-\infty < x < \infty$
- $\frac{1}{x^2+4}$ is continuous for all $-\infty < x < \infty$
- $f'(x) = -(x^2+4)^{-2}\,(x^2+4)' = -(x^2+4)^{-2}\,(2x)$

$$= -\frac{2x}{(x^2+4)^2} < 0 \text{ on } [1, \infty)$$

$$f(x) = \frac{1}{x^2+4} \text{ is decreasing on } [1, \infty)$$

2. Do the Integral Test.



$\lim_{t\to\infty} \arctan t = \frac{\pi}{2}$

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \frac{1}{2}\arctan\frac{x}{2}\Big]_1^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left(\arctan\frac{t}{2} - \arctan\frac{1}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2} - \arctan\frac{1}{2}\right)$$

$\boxed{\int_1^{\infty} \frac{1}{x^2+4}\,dx \text{ is convergent.}}$

$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+4} \text{ is convergent}}$

$\boxed{\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{conv.}, & \text{if } p > 1 \\ \text{div.}, & \text{if } p \le 1 \end{cases}}$

$\int_1^{\infty} \frac{1}{\boxed{x^2}+4}\,dx \qquad p = 2 > 1$

$\boxed{\text{convergent}}$

Example 2. Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent. $(p>0)$

We'll use the Integral Test.

$$a_n = \underbrace{\frac{1}{n^p} = f(n)}, \quad f(x) = \frac{1}{x^p}$$

1. Show that $f(x) = \frac{1}{x^p}$ is positive ✓, continuous ✓, decreasing ✓ on $[1,\infty)$.

- $\frac{1}{x^p} > 0$ on $[1,\infty)$
- $\frac{1}{x^p}$ has discontinuity @ $x=0$. is continuous on $[1,\infty)$
- $f'(x) = -px^{-p-1} = -\frac{p}{x^{p+1}} < 0$ if $p>0$ and $x \geq 1$.

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2. Do the Integral Test.

$$\boxed{\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{convergent, if } p>1 \\ \text{divergent, if } p \leq 1 \end{cases}}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent, if } p>1 \\ \text{divergent, if } p \leq 1 \end{cases}}$$

The Comparison Test I Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series, such that $0 < a_n \le b_n$ for all n.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent

(b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

The Comparison Test II Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and

The Limit Comparison Test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0.$$

Then either both series converge or both diverge.

Example 3. Determine whether the series is convergent or divergent.

(a) $\sum_{n=3}^{\infty} \boxed{\frac{1}{n^2 - 4}}$

Limit Comparison Test.

compare with $\sum_{n=3}^{\infty} \frac{1}{n^2}$ $\quad a_n = \frac{1}{n^2-4}$, $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1 \neq 0, \neq \infty.$$

$\sum_{n=3}^{\infty} \frac{1}{n^2}$ $(p = 2 > 1)$ convergent.

By the limit comparison Test, $\boxed{\sum_{n=3}^{\infty} \frac{1}{n^2-4} \text{ is convergent}}$

(b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

use the original comparison Test.

$$-1 \le \sin n \le 1$$

$$\frac{0}{n\sqrt{n}} \le \frac{\sin^2 n}{n\sqrt{n}} \le \frac{1}{n\sqrt{n}}$$

$$0 \le \frac{\sin^2 n}{n\sqrt{n}} \le \frac{1}{n\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $(p = \frac{3}{2} > 1)$ convergent

By Comparison Test, $\boxed{\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}} \text{ is convergent.}}$

(c) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Limit Comparison Test.

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $\left(p=\frac{1}{2}<1\right)$ – divergent.

$a_n = \frac{1}{\sqrt{n}-1}$, $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}-1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}-1} = 1 \begin{array}{l} \neq 0 \\ \neq \infty \end{array}$$

By the Limit Comparison Test, $\boxed{\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} \text{ is divergent}}$

Estimating the sum of a series

Suppose we've been able to show that a series $\sum_{n=1}^{\infty} a_n$ converges by Integral Test. We want to find an approximation to the sum S of the series. We can approximate S by partial sums S_n. How good is such an approximation?

We need to estimate the size of the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + ...$$

R_n is the error made when the partial sum S_n is used to approximate S.

$$R_n = a_{n+1} + a_{n+2} + ... \leq \int_n^{\infty} f(x)dx$$

here $f(n) = a_n$. Similarly,

$$R_n = a_{n+1} + a_{n+2} + ... \geq \int_{n+1}^{\infty} f(x)dx$$

Remainder estimate for the integral test If $\sum_{n=1}^{\infty} a_n$ converges by the Integral Test, $\sum_{n=1}^{\infty} a_n = s$, $\sum_{k=1}^{n} a_k = s_n$, and $R_n = s - s_n$, then

$$\boxed{\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_{n}^{\infty} f(x)dx}$$

approximation for the remainder of the series.

or

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx}$$

approximation for the sum of the series

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Example 4. (a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ by using the sum of first 5 terms. Estimate the error involved in this approximation.

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4}$$

$$= 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625}$$

Find the approximation for the remainder R_5.

$$\int_6^{\infty} \frac{1}{x^4}dx \le R_5 \le \int_5^{\infty} \frac{1}{x^4}dx$$

$$\boxed{\int_n^{\infty} \frac{1}{x^4}dx \quad (n \geq 1) = \lim_{t\to\infty} \int_n^t x^{-4}dx = \lim_{t\to\infty} \left[\frac{x^{-4+1}}{-4+1}\right]_n^t = \lim_{t\to\infty}\left(\cancel{\frac{t^{-3}}{-3}}^{\,0} - \frac{n^{-3}}{-3}\right) = \frac{1}{3n^3}}$$

$$\boxed{\frac{1}{3(6^3)} \le R_5 \le \frac{1}{3(5^3)}}$$

Error involved in the approximation.

(b) How many terms are required to ensure that the sum is accurate to within 10^{-5}?

$$R_n < 10^{-5}$$

$$\int_{n+1}^{\infty} \frac{1}{x^4}\,dx \le R_n \le \underbrace{\int_{n}^{\infty} \frac{1}{x^4}\,dx}$$

$$\frac{1}{3n^3} < 10^{-5}$$

$$\frac{1}{3n^3} < \frac{1}{10^5}$$

$$3n^3 > 10^5$$

$$n^3 > \frac{10^5}{3}$$

$$n > \sqrt[3]{\frac{10^5}{3}} \approx 32.18$$

$$\boxed{n = 33}$$