Test for divergence. If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

The Integral Test Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

The Comparison Test I Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series, such that $0 < a_n \leq b_n$ for all n.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent

(b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

The Comparison Test II Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0.$$

Then either both series converge or both diverge.

Section 10.4 **Other Convergence Tests**

An alternating series is a series of the form

$$b_1 - b_2 + b_3 - b_4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} b_n,$$

$a_n = (-1)^{n+1} b_n$

$b_n = |a_n|$

where $b_n > 0$ for all n.

The Alternating Series Test. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ satisfies

(a) $\boxed{b_{n+1} \le b_n \text{ for all } n}$ (b) $\boxed{\lim_{n\to\infty} b_n = 0,}$

then the series is convergent.

Example 1. Test the series for convergence or divergence.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^n}$

$a_n = (-1)^{n-1}\frac{n}{2^n}$, $b_n = \left|(-1)^{n-1}\frac{n}{2^n}\right| = \frac{n}{2^n}$

• $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{2^n} = \left|\frac{\infty}{\infty}\right| \underset{=}{\text{L'Hospital's Rule}} \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0$

• Show that $b_{n+1} \le b_n$ for all n.

$b_1 = \frac{1}{2^1} = \frac{1}{2}$; $b_2 = \frac{2}{2^2} = \frac{1}{2}$; $b_3 = \frac{3}{2^3} = \frac{3}{8} \Rightarrow b_1 = b_2 \ge b_3$

Show that $f(n) = \frac{n}{2^n}$ is decreasing.

$f'(n) = (n2^{-n})' = 2^{-n} + n2^{-n}\ln(2)(-1)$

$= 2^{-n}(1 - n\ln 2) < 0$ for all $n \ge 2$.

$(\ln 2 \approx 0.69)$

$\boxed{\text{for all } n,\ b_{n+1} \le b_n}$

$\boxed{\sum_{n=1} (-1)^{n-1} \frac{n}{2^n} \text{ is convergent}}$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{6n-5}$

$a_n = (-1)^n \frac{n}{6n-5}$, $b_n = |a_n| = \left|(-1)^n \frac{n}{6n-5}\right| = \frac{n}{6n-5}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{6n-5} = \frac{1}{6} \ne 0$

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{n}{6n-5} \text{ is divergent}}$

approximating the sum of the series by the n-th partial sum S_n.

$S \approx S_n$, $R_n = S - S_n$

Alternating series estimating theorem. If $s = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is the sum of alternating series that satisfies the Alternating Series Test, then

$$\boxed{|R_n| = |s - s_n| \le b_{n+1}}$$

Example 2. Approximate the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ to four decimal places.

$n! = n(n-1)(n-2) \cdot \ldots \cdot (2)(1)$

$0! = 1$

$$|R_n| \le b_{n+1} = \frac{1}{2^{n+1}(n+1)!} \le 10^{-4}$$

$$2^{n+1}(n+1)! \ge 10^4$$

$n = 4$: $\quad 2^{4+1}(4+1)! = \underbrace{2^5}_{32}\underbrace{(5!)}_{120} = 3840$

$\boxed{n = 5}$: $\quad 2^{5+1}(5+1)! = \underbrace{2^6}_{64}\underbrace{(6!)}_{720} = 46080 > 10^4$

$$S \approx S_5 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$$

$$= 1 + \frac{(-1)}{2} + \frac{1}{2^2(2!)} + \frac{(-1)}{\underbrace{2^3}_{8}\underbrace{(3!)}_{6}} + \frac{1}{\underbrace{2^4}_{16}\underbrace{(4!)}_{24}} + \frac{(-1)}{\underbrace{2^5}_{32}\underbrace{(5!)}_{120}}$$

$$\boxed{S \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \frac{1}{3840}}$$

Given $\sum_{n=1}^{\infty} a_n$, not all a_n are nonnegative.

Definition. a series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent, if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Definition. If $\sum_{n=1}^{\infty} |a_n|$ is divergent, but $\sum_{n=1}^{\infty} a_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

For a series with positive terms only "absolutely convergent" = convergent.

Theorem. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Example 3. Determine whether the series is absolutely convergent.

1. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$

$|n^2| = n^2$

$$\sum_{n=1}^{\infty} \left| \frac{\sin 2n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin 2n|}{n^2}$$

$$\frac{0}{n^2} \le \frac{|\sin 2n|}{n^2} \le \frac{1}{n^2}$$

compare $\sum_{n=1}^{\infty} \frac{|\sin 2n|}{n^2}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ — $\left\{ \begin{matrix} p=2>1 \\ \text{convergent} \end{matrix} \right\}$

By Comparison Test I, $\sum_{n=1}^{\infty} \frac{|\sin 2n|}{n^2}$ is convergent

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2} \text{ is absolutely convergent}}$$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

$2n+1>0$ $(n \ge 1)$
$|2n+1| = 2n+1$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$p=1$
divergent

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ alternating series.

$b_n = \frac{1}{2n+1}$

So the alternating series Test.

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$
- show that $f(n) = (2n+1)^{-1}$ is decreasing.

$$f'(n) = -1(2n+1)^{-2}(2) = -\frac{2}{(2n+1)^2} < 0 \text{ for all } n.$$

Thus, $b_{n+1} \le b_n$ for all n.

By the Alternating series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ is convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ is conditionally convergent.

The Ratio Test. Given a series $\sum_{n=1}^{\infty} a_n$. Let

$$\boxed{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.}$$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$, then the series is divergent
3. If $L = 1$, then the test is inconclusive. (use the Integral or Comparison Tests).

Example 4. Test the series for absolutely convergence, (conditionally) convergence or divergence

1. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2n^2+1}$

$$\lim_{n\to\infty}\frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$$

$\boxed{\text{Divergent}}$ by the Test for Divergence.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ $\boxed{\text{absolutely convergent}}$

$$\lim_{n\to\infty}\frac{1}{n!} = 0$$

Ratio Test.

$$a_n = \frac{1}{n!}\ ; \quad a_{n+1} = \frac{1}{(n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{\cancel{n(n-1)\cdots 2\cdot 1}}{(n+1)\cancel{n(n-1)\cdots 2\cdot 1}}$$

$$= \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{5^n}{n^2}$ $\boxed{\text{divergent}}$

Ratio test

$$a_n = (-1)^{n+1}\frac{5^n}{n^2}, \quad a_{n+1} = (-1)^{n+2}\frac{5^{n+1}}{(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{5^{n+1}}{(n+1)^2}}{(-1)^{n+1}\frac{5^n}{n^2}}\right|$$

$$= \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)^2}\cdot\frac{n^2}{5^n} = \lim_{n\to\infty}\frac{\cancel{5^n}\cdot 5\cdot n^2}{\cancel{5^n}(n+1)^2} = \lim_{n\to\infty}\frac{5n^2}{(n+1)^2} = 5 > 1$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)5^n}{n3^{2n}} = \sum_{n=1}^{\infty} \frac{(n+1)\,5^n}{n\,(3^2)^n} = \sum_{n=1}^{\infty} \frac{(n+1)5^n}{n\,9^n}$ $\boxed{\text{absolutely convergent}}$

Ratio Test.

$$a_n = \frac{(n+1)\,5^n}{n\,9^n}, \quad a_{n+1} = \frac{(\underbrace{(n+1)+1}_{n+2})\,5^{n+1}}{(n+1)\,9^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+2)\,5^{n+1}}{(n+1)\,9^{n+1}}}{\dfrac{(n+1)\,5^n}{n\,9^n}} = \lim_{n\to\infty} \frac{(n+2)\,\cancel{5^{n+1}}}{(n+1)\,\cancel{9^{n+1}}} \cdot \frac{n\,\cancel{9^n}}{(n+1)\,\cancel{5^n}}$$

$$= \lim_{n\to\infty} \frac{n(n+2)\,5}{(n+1)^2\,9} = \frac{5}{9} < 1$$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{n!10^n}$ absolutely convergent.

$$a_n = \frac{(n+2)!}{n!\,10^n} = \frac{(n+2)(n+1)\cancel{n(n-1)\cdots 2\cdot 1}}{\cancel{n(n-1)\cdots 2\cdot 1}\,(10^n)} = \frac{(n+2)(n+1)}{10^n}$$

$$a_{n+1} = \frac{((n+1)+2)\,((n+1)+1)}{10^{n+1}} = \frac{(n+3)(n+2)}{10^{n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+3)\cancel{(n+2)}}{\cancel{10^{n+1}}}}{\dfrac{\cancel{(n+2)}(n+1)}{\cancel{10^n}}} = \lim_{n\to\infty} \frac{(n+3)}{10(n+1)} = \frac{1}{10} < 1$$

Example 5. For which of the following series is the Ratio Test inconclusive?

1. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$a_n = \frac{1}{n^3}, \quad a_{n+1} = \frac{1}{(n+1)^3}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$

inconclusive.

2. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

absolutely conv.

$a_n = \frac{n}{2^n}, \quad a_{n+1} = \frac{n+1}{2^{n+1}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{(n+1)\cancel{2^n}}{\cancel{2^{n+1}}(n)} = \frac{1}{2} < 1$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{1+n^2}$

inconclusive.