Section 10.5 **Power series**

A **power series** is a series of the form

$a_n = c_n x^n$

$$\boxed{\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots}$$

Constants c_n are called the **coefficients** of the series. For each fixed x, the series $\sum_{n=0}^{\infty} c_n x^n$ is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

whose domain is the set of all x for which the series converges.

More generally, a series of the form

$$\boxed{\sum_{n=0}^{\infty} c_n (x-a)^n}$$

is called a **power series centered at a** or a **power series about a**.

A power series is convergent if $|x-a|<R$, where

$$\boxed{R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|}$$

or

$$\boxed{R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}}}$$

R is called the radius of convergence.

- If $R=0$, then the series converges only at one point $x=a$.
- If $R=\infty$, then the series converges for all x.
- If $R\neq 0$ and $R<\infty$, then the series converges if $a-R<x<a+R$. Also, test the series for convergence at $x=a-R$ and $x=a+R$.

The interval of convergence of a power series is the interval that consists of all values of x for which the series is convergent.

Example. Find the radius of convergence and interval of convergence for each of the following series

1. $\sum_{n=0}^{\infty} x^n$

$a_n = x^n$, $a_{n+1} = x^{n+1}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{\cancel{n+1}}}{\cancel{x^n}}\right| = \lim_{n\to\infty} |x|$

$= |x| < 1$

$|x| < 1$

$-1 < x < 1$

$x=-1$: $\sum_{n=0}^{\infty} (-1)^n$ - diverges

$x=1$: $\sum_{n=1}^{\infty} 1^n$ - diverges

interval of convergence $= \boxed{(-1,1)}$

the radius of convergence $= \frac{\text{length of the interval of conv.}}{2}$

$= \frac{1-(-1)}{2} = \boxed{1 = R}$

2. $\sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$

$a_n = \frac{x^n}{(n+2)!}$, $a_{n+1} = \frac{x^{n+1}}{[(n+1)+2]!} = \frac{x^{n+1}}{(n+3)!}$

Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+3)!}}{\frac{x^n}{(n+2)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{\cancel{n+1}}}{(n+3)!} \cdot \frac{(n+2)!}{\cancel{x^n}} \right|$$

$\boxed{(n+1)! = (n+1)\, n!}$

$(n+3)! = (n+3)(n+2)!$

$$= \lim_{n\to\infty} \left| \frac{x\, \cancel{(n+2)!}}{(n+3)\cancel{(n+2)!}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+3} \right| = 0 < 1$$

the limit is less than 1 for all x.

$\boxed{\text{converges for all } x}$

Interval of convergence $\boxed{(-\infty, \infty)}$

Radius of convergence $\boxed{R = \infty}$

2! $\sum_{n=0}^{\infty} \frac{n!\,(x+1)^n}{\sqrt[3]{n+1}}$, $a_n = \frac{n!\,(x+1)^n}{\sqrt[3]{n+1}}$, $a_{n+1} = \frac{(n+1)!\,(x+1)^{n+1}}{\sqrt[3]{n+2}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,(x+1)^{n+1}}{\sqrt[3]{n+2}}}{\frac{n!\,(x+1)^n}{\sqrt[3]{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x+1)^{\cancel{n+1}}}{\sqrt[3]{n+2}} \cdot \frac{\sqrt[3]{n+1}}{n!\,\cancel{(x+1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\cancel{(n!)}\,(x+1)\sqrt[3]{n+1}}{\cancel{n!}\sqrt[3]{n+2}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\sqrt[3]{n+1}\,(x+1)}{\sqrt[3]{n+2}} \right| = \infty > 1$$

the limit is greater that 1 for all x.

The series diverges for all $x \neq -1$

Series is convergent @ $x = -1$ only.

$\boxed{\begin{array}{l}\text{Radius of convergence} = 0 \\ \text{Interval of convergence} = \{-1\}\end{array}}$

2. $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$

$a_n = \frac{x^n}{n+2}$, $a_{n+1} = \frac{x^{n+1}}{n+3}$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+3}}{\frac{x^n}{n+2}}\right| = \lim_{n\to\infty}\left|\frac{x^{\cancel{n+1}}}{n+3}\cdot\frac{n+2}{\cancel{x^n}}\right| = \underbrace{|x|<1}$$

$-1 < x < 1$

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$ – alternating series, $b_n = \frac{1}{n+2}$

Alternating Series Test:

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n+2} = 0$
- $b_n = \frac{1}{n+2}$, $b_{n+1} = \frac{1}{n+3}$

since $n+2 < n+3$, then

$$\underbrace{\frac{1}{n+2}}_{b_n} > \underbrace{\frac{1}{n+3}}_{b_{n+1}}$$

Thus, $b_{n+1} < b_n$ for all n.

$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$ is convergent

$x=1$: $\sum_{n=0}^{\infty}\frac{1}{n+2}$ $(p=1)$ divergent

Interval of convergence $\boxed{[-1,1)}$

Radius of convergence $\boxed{R = \frac{1-(-1)}{2} = 1}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{\sqrt[3]{n}}$, $a_n = \dfrac{(-1)^n (x-2)^n}{\sqrt[3]{n}}$, $a_{n+1} = \dfrac{(-1)^{n+1}(x-2)^{n+1}}{\sqrt[3]{n+1}}$

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{\cancel{n+1}}(x-2)^{\cancel{n+1}}}{\sqrt[3]{n+1}}}{\dfrac{\cancel{(-1)^n}\cancel{(x-2)^n}}{\sqrt[3]{n}}} \right| = \lim_{n\to\infty} \left| \frac{-(x-2)\sqrt[3]{n}}{\sqrt[3]{n+1}} \right| \quad 1 \quad |-1| = 1 \quad |(-1)^n| = 1$$

$$= \underline{|x-2| < 1}$$

$$-1 < x-2 < 1$$

$$2-1 < x < 2+1$$

$$1 < x < 3$$

$x=1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (1-2)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ $p = 1/3 < 1$ divergent

$x=3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3-2)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ alternating series, $b_n = \dfrac{1}{\sqrt[3]{n}} = n^{-1/3} = f(n)$

Alternating Series Test:

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$
- Show that $f(n) = n^{-1/3}$ is decreasing for $n \geq 1$.

  $f'(n) = -1/3\, n^{-1/3-1} = -1/3\, n^{-4/3} = -\dfrac{1}{3}\dfrac{1}{n^{4/3}} < 0$ for all $n \geq 1$.

  Thus, $b_{n+1} < b_n$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is convergent.

Interval of convergence $\boxed{(1, 3]}$

Radius of convergence $\boxed{R = \frac{3-1}{2} = 1}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2(x+3)^n}{10^n}$

$a_n = \dfrac{n^2(x+3)^n}{10^n}$, $\quad a_{n+1} = \dfrac{(n+1)^2(x+3)^{n+1}}{10^{n+1}}$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2(x+3)^{\cancel{n+1}}}{10^{\cancel{n+1}}}}{\frac{n^2\cancel{(x+3)^n}}{\cancel{10^n}}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+3)}{10n^2}\right| = \left|\frac{x+3}{10}\right| < 1$$

$$-1 < \frac{x+3}{10} < 1$$

$$-10 < x+3 < 10$$

$$-10-3 < x < 10-3$$

$$-13 < x < 7$$

$x=-13$: $\displaystyle\sum_{n=0}^{\infty}\frac{n^2(-13+3)^n}{10^n} = \sum_{n=0}^{\infty}\frac{n^2(-10)^n}{10^n} = \sum_{n=0}^{\infty}\frac{n^2\overbrace{(-1)^n\cancel{10^n}}^{(-10)^n}}{\cancel{10^n}}$

$= \displaystyle\sum_{n=0}^{\infty}(-1)^n n^2$ alternating series. $b_n = n^2$.

$\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} n^2 = \infty \neq 0$ divergent

$x=7$: $\displaystyle\sum_{n=0}^{\infty}\frac{n^2\cancel{(7+3)^n}}{\cancel{10^n}} = \sum_{n=0}^{\infty} n^2$

$\displaystyle\lim_{n\to\infty} n^2 = \infty$. divergent.

Interval of convergence $\boxed{(-13, 7)}$

Radius of convergence $\boxed{R = \frac{7-(-13)}{2} = 10}$