Section 10.6. **Representations of functions as a power series**

Geometric series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

Example 1. Find a power series representation for the function and determine the interval of convergence.

$$\boxed{\frac{1}{1-\text{☺}} = \sum_{n=0}^{\infty} (\text{☺})^n, \quad |\text{☺}|<1}$$

1. $$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} \overbrace{(-x)^n}^{(-1)^n x^n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$$

Interval of convergence $\boxed{-1<x<1}$

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2. $$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

Interval of convergence $x^2<1$

$\boxed{-1<x<1}$

2.' $$\frac{1}{4-x^2} = \frac{1}{4\left(1-\frac{x^2}{4}\right)} = \frac{1}{4}\,\frac{1}{1-\frac{x^2}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(x^2)^n}{4^n} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{x^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{x^{2n}}{4^{n+1}}$$

Interval of convergence: $a_n = \frac{x^{2n}}{4^{n+1}}$, $a_{n+1} = \frac{x^{2(n+1)}}{4^{(n+1)+1}} = \frac{x^{2n+2}}{4^{n+2}}$

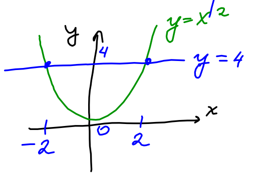
Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{4^{n+2}}}{\frac{x^{2n}}{4^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{x^{\cancel{2n+2}}}{4^{\cancel{n+2}}}\cdot\frac{\cancel{4^{n+1}}}{\cancel{x^{2n}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{4}\right| = \left|\frac{x^2}{4}\right| = \underbrace{\frac{x^2}{4}<1}$$

$x^2<4$

$-2<x<2$



$x=-2$: $\sum_{n=0}^{\infty}\frac{1}{4^{n+1}}\left[(-2)^2\right]^n = \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}4^n = \sum_{n=0}^{\infty}\frac{1}{4}$

divergent, $\lim_{n\to\infty}\frac{1}{4} = \frac{1}{4}\neq 0$

$x=2$: $\sum_{n=0}^{\infty}\frac{1}{4^{n+1}}(2^2)^n = \sum_{n=0}^{\infty}\frac{1}{4^{n+1}}4^n = \sum_{n=0}^{\infty}\frac{1}{4}$ divergent.

Interval of convergence $\boxed{(-2,2)}$

3. $\frac{2}{1+4x^2} = 2 \cdot \frac{1}{1+4x^2} = 2 \cdot \frac{1}{1-(-4x^2)} = 2 \cdot \sum_{n=0}^{\infty} \underbrace{(-4x^2)^n}_{(-1)^n 4^n (x^2)^n}$

$= 2 \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |-4x^2|$

Interval of convergence

$|-4x^2| < 1$

$\frac{4x^2}{4} < \frac{1}{4}$

$x^2 < \frac{1}{4}$

$\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

4. $\dfrac{1+x^2}{1-x^2}$

Improper fraction.

$$\begin{array}{r|l} & -1 \quad \text{whole part} \\ 1-x^2 & 1+x^2 \\ & -(-1+x^2) \\ \hline & 2 \quad \text{remainder} \end{array}$$

$$\frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2}$$

$$= -1 + 2 \cdot \frac{1}{1-x^2}$$

$$= -1 + 2 \sum_{n=0}^{\infty} (x^2)^n$$

$$= \boxed{-1 + 2 \sum_{n=0}^{\infty} x^{2n}}$$

Interval of convergence $x^2 < 1$

$$\boxed{-1 < x < 1}$$

Term-by-term differentiation and integration. If the power series $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ... = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + ... + nc_n(x-a)^{n-1} + ... = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + ... + c_n\frac{(x-a)^{n+1}}{n+1} + ... =$$

$$C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of these series are R. This does not mean that the interval of convergence remains the same.

Example 2. Find a power series representation for the function and determine the radius of convergence.

1. $\dfrac{1}{(1+x)^2}$

$$\left(\frac{1}{1+x}\right)' = -\frac{1}{(1+x)^2} \Rightarrow \frac{1}{(1+x)^2} = -\left(\frac{1}{1+x}\right)'$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x|<1$$

$$\left(\frac{1}{1+x}\right)' = \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)'$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^n)'$$

$$\left(\frac{1}{1+x}\right)' = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

$$\frac{1}{(1+x^2)} = -\left(\frac{1}{1+x}\right)' = -\sum_{n\geq 1}^{\infty} (-1)^n n x^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

Interval of convergence $\boxed{(-1,1)}$

2. $\ln(4+x) = \int \frac{1}{4+x}\,dx$

$$\frac{1}{4+x} = \frac{1}{4\left(1+\frac{x}{4}\right)} = \frac{1}{4}\,\frac{1}{1+\frac{x}{4}} = \frac{1}{4}\,\frac{1}{1-\left(-\frac{x}{4}\right)}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\,\frac{x^n}{4^n}$$

$$\left(-\frac{x}{4}\right)^n$$

$$\ln(4+x) = \int\frac{1}{4+x}\,dx = \frac{1}{4}\int\left(\sum_{n=0}^{\infty}(-1)^n\,\frac{x^n}{4^n}\right)dx.$$

Interval of convergence: $\left|\frac{x}{4}\right|<1$ or $|x|<4$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\,\frac{1}{4^n}\left(\int x^n\,dx\right)$$

$$\ln(4+x) = C + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\,\frac{1}{4^n}\,\frac{x^{n+1}}{n+1}$$

Plug $x=0$: $\ln 4 = C + \frac{1}{4}(0) \Rightarrow C = \ln 4$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p} \text{ converges, if } p>0$$

$$\boxed{\ln(4+x) = \ln 4 + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\,\frac{1}{4^n}\,\frac{x^{n+1}}{n+1}}$$

Interval of convergence: $\boxed{(-4,4]}$

$x=4$: $\frac{1}{4}\sum_{n=0}^{\infty}\frac{4(-1)^n}{n+1}$ – convergent

Example 3. Evaluate an indefinite integral $\int \tan^{-1}(x^2)\,dx$ as a power series.

$\tan^{-1}(x^2)$ | $u = x^2$, $du = 2x\,dx$

$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$

$\tan^{-1}(u) = \int \frac{1}{1+u^2}\,du \Rightarrow \tan^{-1}(x^2) = \int \frac{2x\,dx}{1+(x^2)^2}$

$$\tan^{-1}(x^2) = \int \frac{2x\,dx}{1+x^4}$$

$$\frac{2x}{1+x^4} = \frac{2x}{1-(-x^4)} = 2x\sum_{n=0}^{\infty}(-x^4)^n$$

$$= 2x\sum_{n=0}^{\infty}\underbrace{(-1)^n x^{4n}}_{(-x^4)^n} = 2\sum_{n=0}^{\infty}(-1)^n x(x^{4n})$$

$$\boxed{\frac{2x}{1+x^4} = 2\sum_{n=0}^{\infty}(-1)^n x^{4n+1}}, \quad |x|<1$$

$$\tan^{-1}(x^2) = \int \frac{2x}{1+x^4}\,dx = 2\int\left(\sum_{n=0}^{\infty}(-1)^n x^{4n+1}\right)dx$$

$$= 2\sum_{n=0}^{\infty}(-1)^n\left(\int x^{4n+1}\,dx\right)$$

$$\tan^{-1}(x^2) = 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2} + C$$

plug $x=0$: $\tan^{-1}(0) = 2(0)+C$

$\boxed{0=C}$

$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p} \text{ convergent, if } p>0}$

$$\boxed{\tan^{-1}(x^2) = 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2}}$$

Interval of convergence: $\boxed{[-1,1]}$

Plug $x=1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+2}$ convergent.

$x=-1$: $\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{4n+2}}{4n+2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+2}$ convergent

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$$\int \tan^{-1}(x^2)\,dx = 2\int\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2}\right)dx$$

$$= 2\sum_{n=0}^{\infty}(-1)^n\frac{1}{4n+2}\left(\int x^{4n+2}\,dx\right)$$

$$\boxed{\int \tan^{-1}(x^2)\,dx = C + 2\sum_{n=0}^{\infty}(-1)^n\frac{1}{4n+2}\frac{x^{4n+3}}{4n+3}} \quad \boxed{[-1,1]}$$

Example 4. Use a power series to approximate the integral

$$\int_0^{1/2} \tan^{-1}(x^2)\, dx$$

to six decimal places.

$$\int \tan^{-1}(x^2)\, dx = C + 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)(4n+3)} x^{4n+3}$$

$$\int_0^{1/2} \tan^{-1}(x^2)\, dx = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)(4n+3)} x^{4n+3} \Big]_0^{1/2}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)(4n+3)} \left(\frac{1}{2}\right)^{4n+3}$$

$$\approx 2\left(\underbrace{\frac{1}{6}\left(\frac{1}{2}\right)^3}_{n=0} - \underbrace{\frac{1}{42}\left(\frac{1}{2}\right)^7}_{n=1} + \underbrace{\frac{1}{110}\left(\frac{1}{2}\right)^{11}}_{n=2} \right)$$