Section 10.7 **Taylor and Maclaurin series**

Let f be any function that can be represented by a power series

$$\boxed{f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots (|x-a| < R)}$$

Let us try to determine coefficients c_n, $n = 0, 1, 2, \ldots$.

$$\boxed{c_0 = f(a)}$$

We can differentiate the series for f term-by-term.

$$f'(x) = c_1 + 2c_2(x-a) + \ldots + nc_n(x-a)^{n-1} + \ldots$$

$$\boxed{c_1 = f'(a)}$$

$$f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + \ldots + n(n-1)(x-a)^{n-2} + \ldots$$

$$\boxed{c_2 = \frac{f''(a)}{2}}$$

$$f'''(x) = 3 \cdot 2c_3 + \ldots + n(n-1)(n-2)(x-a)^{n-3} + \ldots$$

$$\boxed{c_3 = \frac{f'''(a)}{3 \cdot 2} = \frac{f'''(a)}{3!}}$$

So,

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}.}$$

$n!$

Theorem. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R,$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

the series is called the **Taylor series of the function f at a.**

Example 1. Find the Taylor series for the function $f(x) = \dfrac{1}{x}$ at $a = 1$.

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = (-1)x^{-2}, \quad f''(x) = (-1)(-2)x^{-3} = (-1)^2\, 2!\, x^{-3}$$

$$f'''(x) = (-1)(-2)(-3)x^{-4} = (-1)^3\, 3!\, x^{-4}$$

$$\cdots$$

$$f^{(n)}(x) = (-1)^n\, n!\, x^{-(n+1)}$$

$$f^{(n)}(1) = (-1)^n\, n!$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}}(x-1)^n$$

$$\boxed{\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

If we plug 0 for a in the Taylor series, we'll get a series

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

which is called the Maclauren series.

Suppose that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Let

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

T_n is called the n**th-degree Taylor polynomial of** f **at** a .

In general, $f(x)$ is the sum of its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$.

If we let $R_n(x)$ be the remainder of the series, then

$$R_n(x) = f(x) - T_n(x)$$

If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that $\lim_{n\to\infty} T_n(x) = f(x)$. For trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function f, we usually use the following fact.

Taylor's Inequality. If $|f^{(n+1)}(x)| \leq M$, then

$$|R_n| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

Important Maclaurin series and their intervals of convergence.

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (-1, 1)}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (-\infty, \infty)}$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty, \infty)}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad (-\infty, \infty)}$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots, \quad [-1, 1] = \boxed{\sum_{n=0}^{\infty} \frac{m!}{n!\,(m-n)!} x^n}$$

Example 2. Find the Maclaurin series for $f(x) = x^2 \cos(x^3)$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

$$x^2\cos(x^3) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^2 x^{6n}}{(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(2n)!}}$$

E.2* Find the sum of the series:

(a) $\sum_{n=2}^{\infty} \frac{3^n}{n!} = e^3 - 1 - 3$

$= \boxed{e^3 - 4}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

$$e^x = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - 1 - x$$

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{6^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n+1} \frac{1}{(2n+1)!} = \sin \frac{\pi}{6} = \boxed{\frac{1}{2}}$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(2n)!} = \cos \frac{\pi}{4} = \boxed{\frac{\sqrt{2}}{2}}$

Example 3. Use series to evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}.$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = (-1)^0 \frac{x^0}{0!} + (-1)^1 \frac{x^2}{2!} + (-1)^2 \frac{x^4}{4!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\cancel{1} - (\cancel{1} - \frac{x^2}{2} + \frac{x^4}{24} + \dots)}{\cancel{1}+\cancel{x} - (\cancel{1}+\cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + \dots)} = \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^4}{24} + \dots}{-\frac{x^2}{2} - \frac{x^3}{6} - \dots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(\frac{1}{2} - \frac{x^2}{24} + \dots)}{\cancel{x^2}(-\frac{1}{2} - \frac{x}{6} - \dots)} = \frac{\frac{1}{2}}{-\frac{1}{2}} = \boxed{-1}$$

Example 4. Find the Maclaurin series for $\ln(1+x)$ and use it to calculate $\ln 1.1$ correct to five decimal places.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n\right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\int x^n\,dx\right)$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

Plug $x=0$: $\ln 1 = 0 + C \Rightarrow C = \ln 1 = 0$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

$$\ln(1.1) = \ln(1+.1) = \sum_{n=0}^{\infty} (-1)^n \frac{(.1)^{n+1}}{n+1}$$

$$= \frac{.1}{1} - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} - \frac{(.1)^4}{4} + \frac{(.1)^5}{5}$$

$$= .1 - \frac{.01}{2} + \frac{.001}{3} - \frac{.0001}{4} + \frac{.00001}{5}$$

$$= .1 - 0.005 + 0.000333 - 0.000025 + 0.000002$$

$$\approx 0.09531$$

Example 5. Use series to approximate the definite integral $\int_0^{0.05} \cos(x^2)\, dx$ correct to three decimal places.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$\int_0^{0.05} \cos(x^2)\, dx = \int_0^{0.05} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(\int_0^{0.05} x^{4n}\, dx \right)$$

$\boxed{0! = 1}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left. \frac{x^{4n+1}}{4n+1} \right]_0^{0.05}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \frac{(0.05)^{4n+1}}{4n+1}$$

$$\approx (-1)^0 \frac{1}{0!} \frac{0.05}{1} + (-1)^1 \frac{1}{2!} \frac{(0.05)^5}{5}$$

$$= \boxed{0.05 - \frac{(0.05)^5}{10}}$$