Section 10.9 **Applications of Taylor polynomials**

Suppose that

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Consider

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the nth-degree Taylor polynomial of f at a.

We can use a Taylor polynomial T_n to approximate f. But how good an approximation is? To answer this question we need to look at

$$|R_n|=|f(x)-T_n(x)|$$

(a) If the series happen to be an alternating series, then

$$|R_n|\le\frac{|f^{(n+1)}(a)|}{(n+1)!}|x-a|^{n+1}$$

(b) In other cases we can use **Taylor's Inequality**, which says if $|f^{(n+1)}(x)|\le M$, then

$$|R_n|\le\frac{M}{(n+1)!}|x-a|^{n+1}$$

Example 1.
(a) Approximate $f(x)=\sqrt{x}$ by a Taylor polynomial of degree 3 at $a=1$.

(b) How accurate is this approximation if $0.9\le x\le 1.1$?

Example 2. In Eistein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object when at rest and c is the speed of the light ($c \approx 300 \times 10^6$ m/s). The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expression for K when $|v| \leq 100$ m/s.

Example 3. An electric dipole consists of two electric charges of equal magnitude and opposite signs. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.