Section 10.9 **Applications of Taylor polynomials**

Suppose that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Consider the n-th partial sum of the series

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the nth-degree Taylor polynomial of f at a.

$f(x) \approx T_n(x)$

We can use a Taylor polynomial T_n to approximate f. But how good an approximation is? To answer this question we need to look at

$$|R_n| = |f(x) - T_n(x)|$$ – remainder of the series

(a) If the series happen to be an alternating series, then

$$\boxed{|R_n| \le \frac{|f^{(n+1)}(a)|}{(n+1)!}|x-a|^{n+1}}$$

(b) In other cases we can use **Taylor's Inequality**, which says if $|f^{(n+1)}(x)| \le M$, then

$$\boxed{|R_n| \le \frac{M}{(n+1)!}|x-a|^{n+1}}$$

Example 1.

(a) Approximate $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 3 at $a = 1$.

$$\sqrt{x} = f(x) \approx \cancel{f(1)}^{1} + \cancel{f'(1)}^{1/2}(x-1) + \frac{\cancel{f''(1)}^{-1/4}}{2}(x-1)^2 + \frac{\cancel{f'''(1)}^{3/8}}{3!}(x-1)^3$$

$$f(x) = \sqrt{x} \Rightarrow \boxed{f(1) = 1}$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow \boxed{f'(1) = 1/2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}x^{-3/2} \Rightarrow \boxed{f''(1) = -\frac{1}{4}}$$

$$f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right)x^{-5/2} = \frac{3}{8}x^{-5/2} \Rightarrow \boxed{f'''(1) = \frac{3}{8}}$$

$$\sqrt{x} = f(x) \approx 1 + \frac{1}{2}(x-1) - \frac{1}{4(2)}(x-1)^2 + \frac{3}{8(3)!}(x-1)^3 = \boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$$

not alternating

(b) How accurate is this approximation if $0.9 \le x \le 1.1$?

Taylor's Inequality: $|R_3| \le \frac{M}{(3+1)!}|x-1|^{3+1}$, where $|f^{(4)}(x)| \le M$

$$|R_3| \le \frac{M}{4!}|x-1|^4$$

$$0.9 - 1 \le x - 1 \le 1.1 - 1 \Rightarrow -0.1 \le x-1 \le 0.1 \Rightarrow \boxed{|x-1| \le 0.1}$$

Find M. $\quad f^{(4)}(x) = \frac{3}{8}\left(-\frac{5}{2}\right)x^{-7/2} = -\frac{15}{16}x^{-7/2}$

$$|f^{(4)}(x)| = \left|-\frac{15}{16}x^{-7/2}\right| = \left|\frac{15}{16x^{7/2}}\right| \le \boxed{\frac{15}{16(0.9)^{7/2}} = M}$$

$$|R_3| \le \boxed{\frac{15}{16(0.9)^{7/2}(24)}(0.1)^4} \approx 5.648 \times 10^{-6}$$

Example 2. In Eistein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

where m_0 is the mass of the object when at rest and c is the speed of the light ($c \approx 300 \times 10^6$ m/s). The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2 \quad \text{- kinetic energy}$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

v is very small compared with c.

$$K = mc^2 - m_0c^2 = \underbrace{m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}}_{m} c^2 - m_0c^2$$

$$= m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \dots$, plug $m = -1/2$, $x = -\frac{v^2}{c^2}$

$$= m_0c^2\left[\cancel{1} + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \cancel{\frac{-1/2(-1/2-1)}{2}\left(-\frac{v^2}{c^2}\right)^2}^{\,0} + \dots - \cancel{1}\right]$$

v is very small compared with $\Rightarrow \left(\frac{v^2}{c^2}\right)^k \approx 0$, $k \geq 2$

$$K \approx m_0\cancel{c^2}\,\frac{1}{2}\,\frac{v^2}{\cancel{c^2}} = \frac{1}{2}m_0v^2$$

(b) Use Taylor's Inequality to estimate the difference in these expression for K when $|v| \leq 100$ m/s.

$K = mc^2 - m_0c^2 \approx T_1(x)$, where $x = -\frac{v^2}{c^2}$, $f(x) = (1+x)^{-1/2}$

$$|R_1| \leq \frac{M}{2!}|x|^2$$

$$|R_1| \leq \frac{M}{2}\left|\frac{v^2}{c^2}\right|^2, \quad \text{where} \quad |f''(x)| \leq M$$

$$f'(x) = -\frac{1}{2}(1+x)^{-3/2}$$

$$f''(x) = -\frac{1}{2}\left(-\frac{3}{2}\right)(1+x)^{-5/2} = \frac{3}{4}(1+x)^{-5/2}$$

$$x = -\frac{v^2}{c^2} \Rightarrow |x| = \frac{v^2}{c^2} \leq \frac{(100)^2}{(3\times10^8)^2} = \frac{1}{9\times10^{12}}$$

$$|x| \leq \frac{1}{9\times10^{12}}$$

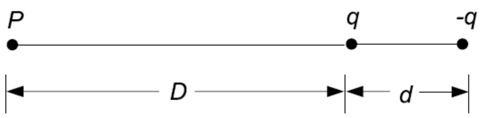
$$|f''(x)| \leq \boxed{\frac{3}{4}\left(1+\frac{1}{9\times10^{12}}\right)^{-5/2} = M}$$

$$\boxed{|R_1| \leq \frac{1}{2}\underbrace{\frac{3}{4}\left(1+\frac{1}{9\times10^{12}}\right)^{-5/2}}_{M}\underbrace{\left(\frac{1}{9\times10^{12}}\right)^2}_{\left|\frac{v^2}{c^2}\right|^2}}$$

Example 3. An electric dipole consists of two electric charges of equal magnitude and opposite signs. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole.



$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2} - q(D+d)^{-2}$$

$$= \frac{q}{D^2} - q\left[D\left(1+\frac{d}{D}\right)\right]^{-2} = \frac{q}{D^2} - q\left[D^{-2}\left(1+\frac{d}{D}\right)^{-2}\right]$$

$$= \frac{q}{D^2} - \frac{q}{D^2}\left(1+\frac{d}{D}\right)^{-2} = \frac{q}{D^2}\left[1-\left(1+\frac{d}{D}\right)^{-2}\right]$$

$$\boxed{(1+x)^m = 1+mx+\frac{m(m-1)}{2}x^2+\frac{m(m-1)(m-2)}{6}x^3+\dots}$$

use it for $m=-2$ and $x=\frac{d}{D}$

$$\left(1+\frac{d}{D}\right)^{-2} = 1+(-2)\left(\frac{d}{D}\right)+\frac{(-2)(-3)}{2}\left(\frac{d}{D}\right)^2+\dots$$

$$= 1-2\frac{d}{D}+3\left(\frac{d}{D}\right)^2+\dots$$

$$E = \frac{q}{D^2}\left[1-\left(1-2\frac{d}{D}+3\left(\frac{d}{D}\right)^2+\dots\right)\right]$$

$$= \frac{q}{D^2}\left(\cancel{1}-\cancel{1}+2\frac{d}{D}-3\left(\frac{d}{D}\right)^2+\dots\right) = \frac{q}{D^2}\left(2\frac{d}{D}-3\left(\frac{d}{D}\right)^2+\dots\right)$$

$$= \frac{q}{D^2}\,\frac{1}{D}\left(2d-\boxed{3\frac{d^2}{D}+\dots}\right)$$

neglect

if D is very large, we can neglect $\frac{1}{D}, \frac{1}{D^2}, \frac{1}{D^3}, \dots$

$$\boxed{E \approx \frac{2qd}{D^3} \text{ which is proportional to } \frac{1}{D^3}}$$