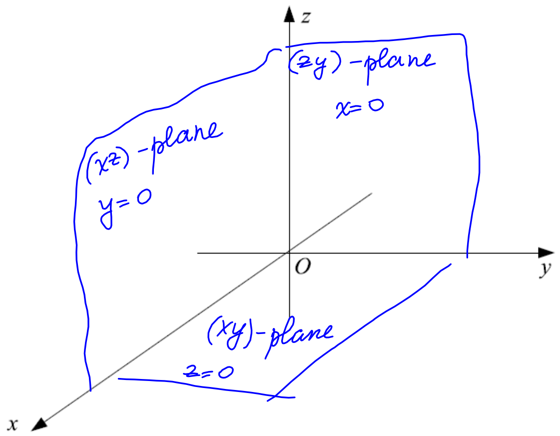
Section 11.1 **Three-dimensional coordinate system**

To locate a point in space three numbers are required. We represent any point in space by an ordered triple (a, b, c).

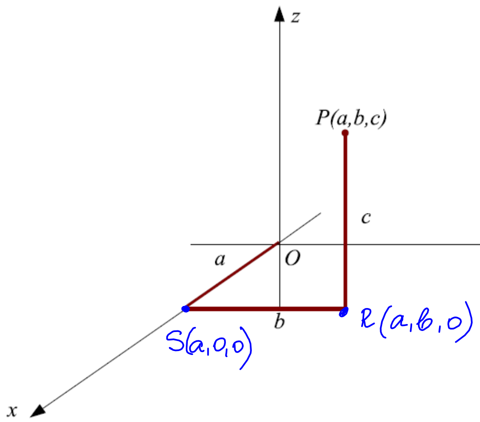
In order to represent points in space, we first choose a fixed point O (the origin) and tree directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x and y-axes as being horizontal and z-axis as being vertical.

The direction of z-axis is determined by the **right-hand rule**: if your index finger points in the positive direction of the x-axis, middle finger points in the positive direction of the y-axis, then your thumb points in the positive direction of the z-axis.



The three coordinate axes determine the three **coordinate planes**. The xy-plane contains the x- and y-axes and its equation is $z = 0$, the xz-plane contains the x- and z-axes and its equation is $y = 0$, The yz-plane contains the y- and z-axes and its equation is $x = 0$. These three coordinate planes divide space into eight parts called **octants**. The **first octant** is determined by positive axes.

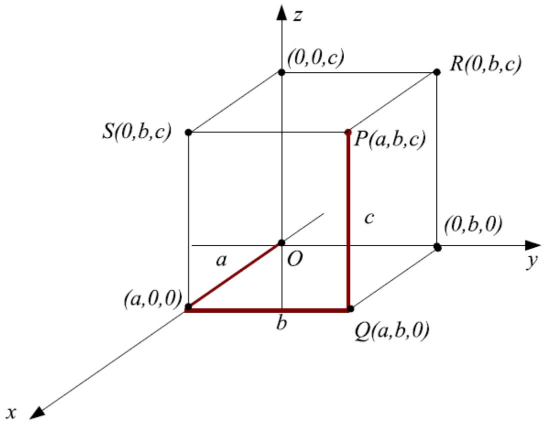
Take a point P in space, let a be directed distance from yz-plane to P, b be directed distance from xz-plane to P, and c be directed distance from xy-plane to P.



$R(a,b,0)$ is the projection of the point $P(a,b,c)$ onto the (xy)-plane.

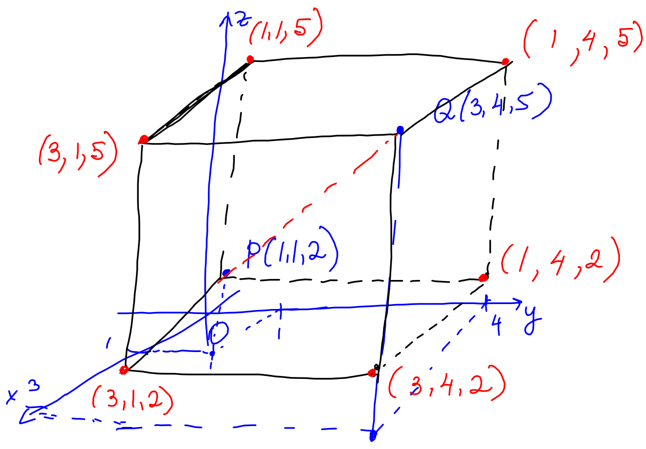
We represent the point P by the ordered triple (a, b, c) of real numbers, and we call a, b, and c the coordinates of P.

The point $P(a, b, c)$ determine a rectangular box.



If we drop a perpendicular from P to the xy-plane, we get a point $Q(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P on the yz-plane and xz-plane, respectively.

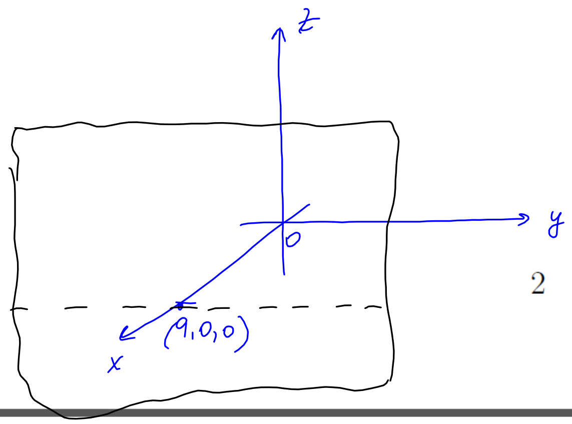
Example 1. Draw a rectangular box that has $P(1, 1, 2)$ and $Q(3, 4, 5)$ as opposite vertices and has its faces parallel to the coordinate planes. Then find the coordinates of the other six vertices of the box.



The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triplets of real numbers. We have given a one-to-one correspondence between points P in space and ordered triplets (a, b, c) in $\mathbb{R}^3$. It is called a tree-dimensional rectangular coordinate system.

Example 2. What surfaces in $\mathbb{R}^3$ represented by the following equations?

1. $x = 9$ – plane, parallel to (yz)-plane, passes through $(9,0,0)$.
Power of x is 1, we are missing y and z.



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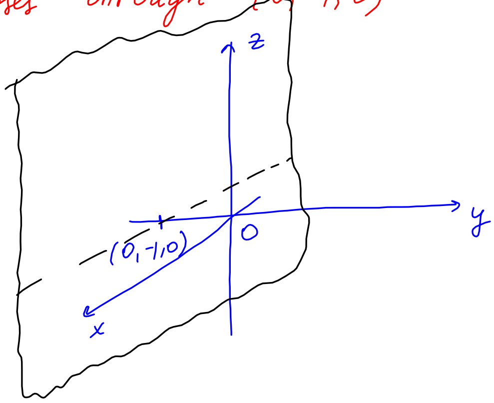
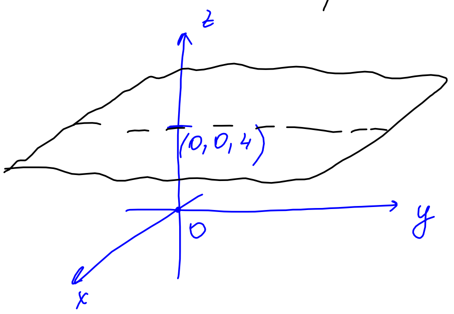
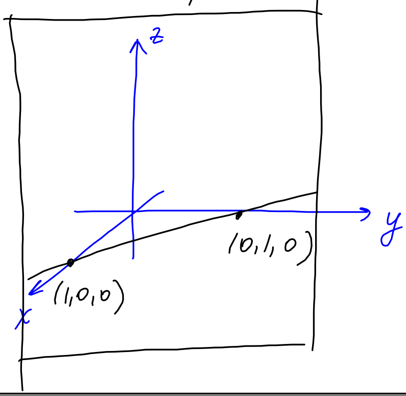
2. $y = -1$ -plane (power of y is 1)
-parallel to the (xz)-plane (x and z are missing)
-passes through $(0,-1,0)$



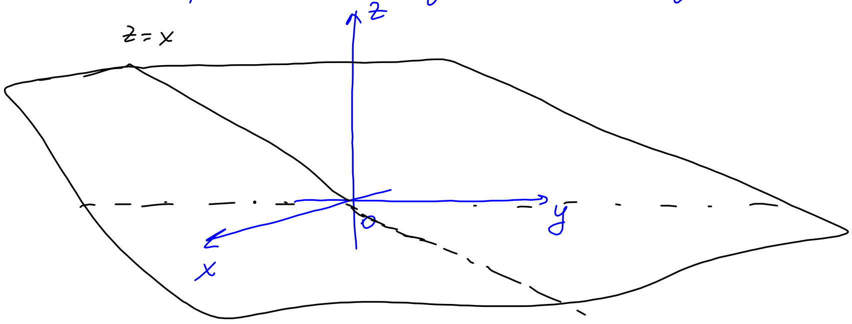
3. $z=4$ - plane, parallel to the (xy)-plane (horizontal), passes through (0,0,4).



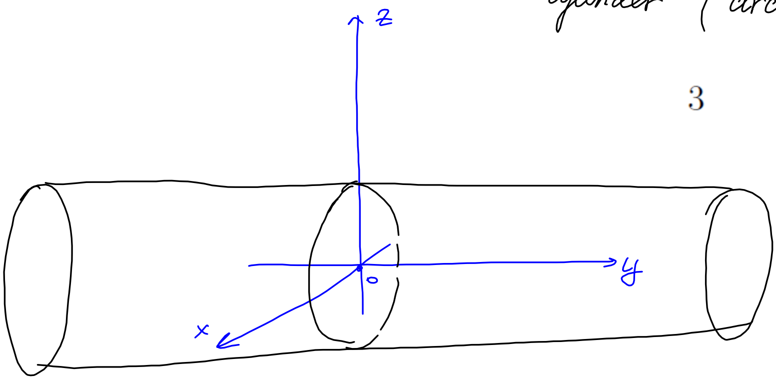
4. $x+y=1$ - plane (powers for x and y =1) parallel to the z axis (missing the z-coordinate)



5. $z = x$ - plane, parallel to the y-axis
passes through the origin



6. $x^2 + z^2 = 9$ – parallel to the y-axis.
cylinder (circular cylinder).



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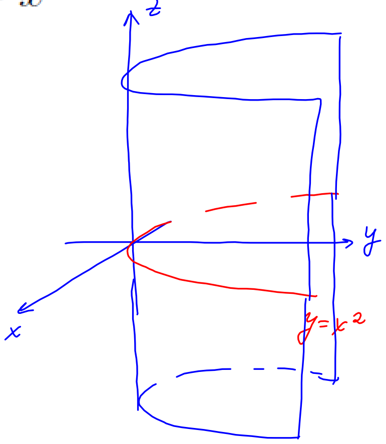
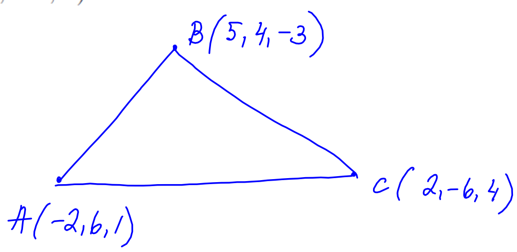
7. $y = x^2$ - cylinder (parabolic cylinder), parallel to the z-axis.



The distance formula in three dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

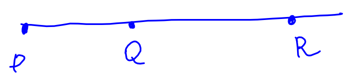
$$\boxed{|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Example 3. Find the length of the sides of the triangle ABC, where $A(-2, 6, 1)$, $B(5, 4, -3)$, and $C(2, -6, 4)$.



$$|BC| = \sqrt{(2-5)^2 + (-6-4)^2 + (4-(-3))^2}$$
$$= \sqrt{9+100+49} = \sqrt{158}$$

Example 4. Determine whether the points $P(1, 2, 3)$, $Q(0, 3, 7)$, and $R(3, 5, 11)$ are $\underbrace{\text{collinear}}_{\text{lying on the same line.}}$

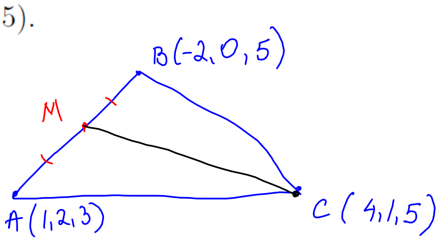


if $|PR| = |PQ| + |QR| \Rightarrow P, Q, R$ are collinear.

The **midpoint of the line segment** from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\boxed{P_M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)}$$

Example 3. Find the length of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.



CM is a median

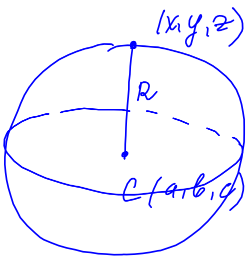
$$M\left(\frac{-2+1}{2}, \frac{0+2}{2}, \frac{5+3}{2}\right)$$

$$= M\left(-\frac{1}{2}, 1, 4\right)$$

$$|CM| = \sqrt{\left(-\frac{1}{2}-4\right)^2 + (1-1)^2 + (4-5)^2}$$

$$= \sqrt{\left(-\frac{9}{2}\right)^2 + 1} = \sqrt{\frac{81}{4}+1} = \sqrt{\frac{85}{4}} = \boxed{\frac{\sqrt{85}}{2}}$$

Definition. A sphere is the set of all points that are equidistant from the center.
Problem Find an equation of a sphere of radius R and center $C(a, b, c)$.



Let $P(x,y,z)$ be an arbitrary point on the sphere.

$|CP|^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

$|CP| = R$

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2}$$

Equation of a sphere centered at (a,b,c) of radius R.

Equation of a sphere of radius R and center $C(a, b, c)$ is

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2}$$

Example 5. Find an equation of a sphere of radius $R = 4$ centered at $C(\underbrace{-1}_{a}, \overbrace{2}^{b}, \underbrace{4}_{c})$.

$$(x-(-1))^2 + (y-2)^2 + (z-4)^2 = 4^2 \Rightarrow \boxed{(x+1)^2 + (y-2)^2 + (z-4)^2 = 16}$$

Example 6. Find radius and center of sphere given by the equation

$$x^2 + y^2 + z^2 + x - 2y + 6z - 2 = 0$$

Complete squares.

$$(x^2+x) + (y^2-2y) + (z^2+6z) - 2 = 0.$$

$$\left(x^2+x+\tfrac{1}{4}\right) + (y^2-2y+1) + (z^2+6z+9) - \tfrac{1}{4} - 1 - 9 - 2 = 0$$

$$\left(x+\tfrac{1}{2}\right)^2 + (y-1)^2 + (z+3)^2 - \frac{49}{4} = 0$$

$$\left(x+\tfrac{1}{2}\right)^2 + (y-1)^2 + (z+3)^2 = \frac{49}{4}$$

$$\boxed{R = \tfrac{7}{2};\quad C\left(-\tfrac{1}{2}, 1, -3\right)}$$

Example 7. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

$P(x, y, z)$

$$(|PA|)^2 = (2|PB|)^2 \Rightarrow |PA|^2 = 4|PB|^2$$

$$|PA| = \sqrt{(-1-x)^2 + (5-y)^2 + (3-z)^2}$$

$$|PB| = \sqrt{(6-x)^2 + (2-y)^2 + (-2-z)^2}$$

$$(-1-x)^2 + (5-y)^2 + (3-z)^2 = 4\left((6-x)^2 + (2-y)^2 + (-2-z)^2\right)$$

$$1 + 2x + x^2 + 25 - 10y + y^2 + 9 - 6z + z^2 = 4\left(36 - 12x + x^2 + 4 - 4y + y^2 + 4 + 4z + z^2\right)$$

$$35 + 2x + x^2 - 10y + y^2 - 6z + z^2 = 176 - 48x + 4x^2 - 16y + 4y^2 + 16z + 4z^2$$

$$4x^2 - 48x - 2x - x^2 + 4y^2 - 16y - y^2 + 10y + 4z^2 + 16z - z^2 + 6z = 35 - 176$$

$$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = -141$$

$$\frac{3\left(x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z\right)}{3} = \frac{-141}{3}$$

$$\left[x^2 - \frac{50}{3}x + \left(\frac{25}{3}\right)^2\right] + (y^2 - 2y + 1) + \left[z^2 + \frac{22}{3}z + \left(\frac{11}{3}\right)^2\right] - \frac{625}{9} - 1 - \frac{121}{9} = -\frac{141}{3}$$

$$\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \underbrace{-\frac{141}{3} + \frac{755}{9}}$$

$$\boxed{\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{332}{9}}$$

sphere of radius $\dfrac{\sqrt{332}}{3}$

centered @ $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$