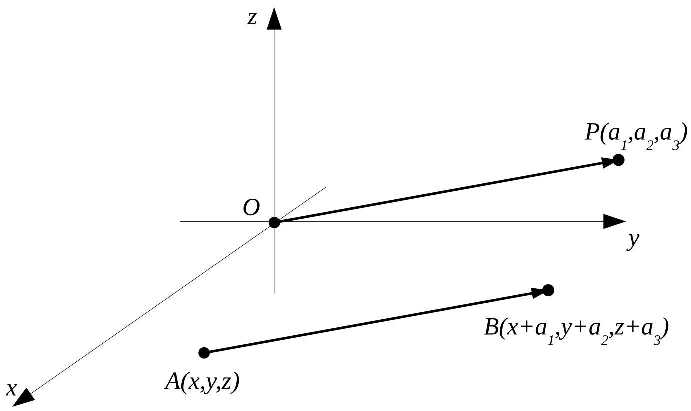
Section 11.2 **Vectors and the dot product in three dimensions**

Geometrically, a three-dimensional vector can be considered as an arrow with both a length and direction. An arrow is a directed line segment with a starting point and an ending point. Algebraically, a **tree-dimensional vector** is an ordered triple $\vec{a} =< a_1, a_2, a_3 >$ of real numbers. The numbers a_1, a_2, and a_3 are called the **components** of $\vec{a}$.

A **representation** of the vector $\vec{a} =< a_1, a_2, a_3 >$ is a directed line segment $\vec{AB}$ from any point $A(x, y, z)$ to the point $B(x + a_1, y + a_2, z + a_3)$.

A particular representation of $\vec{a} =< a_1, a_2, a_3 >$ is the directed line segment $\vec{OP}$ from the origin to the point $P(a_1, a_2, a_3)$, and $\vec{a} =< a_1, a_2, a_3 >$ is called the **position vector** of the point $P(a_1, a_2, a_3)$.



Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then $\boxed{\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >}$.

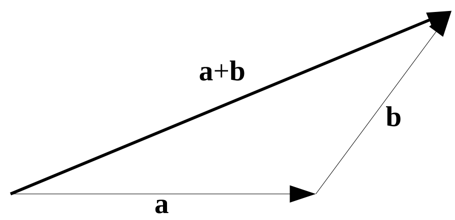
Example 1. Find a vector $\vec{a}$ with representation given by the directed line segment $\vec{AB}$, where $A(1, -2, 0)$, $B(1, -2, 3)$. Draw $\vec{AB}$ and the equivalent representation starting at the origin.

The **magnitude (length)** $|\vec{a}|$ of $\vec{a}$ is the length of any its representation.

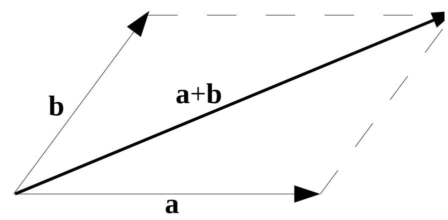
The length of $\vec{a}$ is $\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$

The only vector with length 0 is the **zero vector** $\vec{0} =< 0, 0, 0 >$. This vector is the only vector with no specific direction.

Vector addition If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then the vector $\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$



Triangle Law

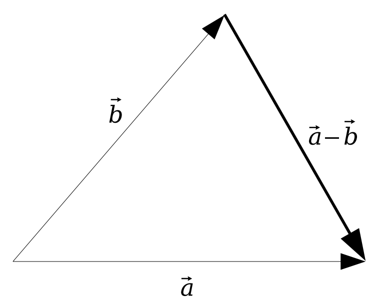


Parallelogram Law

Multiplication of a vector by a scalar If c is a scalar and $\vec{a} =< a_1, a_2, a_3 >$, then the vector $c\vec{a} =< ca_1, ca_2, ca_3 >$.

Two vectors $\vec{a}$ and $\vec{b}$ are called **parallel** if $\vec{b} = c\vec{a}$ for some scalar c. If $\vec{a} =< a_1, a_2, a_3 >$, $\vec{b} =< b_1, b_2, b_3 >$, then $\vec{a}$ and $\vec{b}$ are parallel if and only if $\boxed{\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}}$.

By the **difference** of two vectors, we mean $\boxed{\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >}$



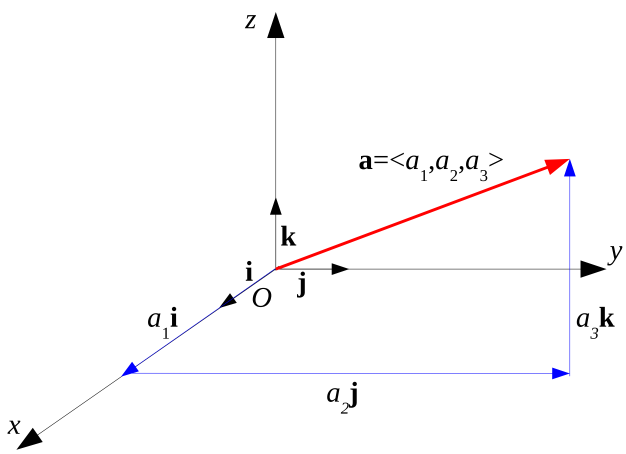
Example 2. Find $|\vec{a}|$, $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, $2\vec{a} - 5\vec{b}$ if $\vec{a} =< 1, -3, 2 >$, $\vec{b} =< 2, 1, -1 >$.

Properties of vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k and m are scalars, then

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$
6. $(k+m)\vec{a}=k\vec{a}+m\vec{a}$
7. $(km)\vec{a}=k(m\vec{a})$
8. $1\vec{a}=\vec{a}$

Let $\vec{\imath}=<1,0,0>$ and $\vec{\jmath}=<0,1,0>$, $\vec{k}=<0,0,1>$, $|\vec{\imath}|=|\vec{\jmath}|=|\vec{k}|=1$.

$$\vec{a}=<a_1,a_2,a_3>=a_1\vec{\imath}+a_2\vec{\jmath}+a_3\vec{k}$$



A **unit vector** is a vector whose length is 1.

A vector $\boxed{\vec{u}=\dfrac{1}{|\vec{a}|}\vec{a}=\left\langle\dfrac{a_1}{|\vec{a}|},\dfrac{a_2}{|\vec{a}|},\dfrac{a_3}{|\vec{a}|}\right\rangle}$ is a unit vector that has the same direction as $\vec{a}=<a_1,a_2,a_3>$.

Example 3. Find the unit vector in the direction of the vector $\vec{\imath}-2\vec{\jmath}+2\vec{k}$.

Definition. The **dot** or **scalar product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number $\boxed{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a}\cdot\vec{b}=0$.

$\vec{a} \cdot \vec{b} > 0$ if and only if $0 < \theta < \pi/2$
$\vec{a} \cdot \vec{b} < 0$ if and only if $\pi/2 < \theta < \pi$

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then $\boxed{\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$.

$$\boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}}$$

Example 4. Given $\vec{a} =< 2, 3, -4 >$, $\vec{b} =< 1, -4, 8 >$. Find $\vec{a} \cdot \vec{b}$.

Example 5. Find the angle between vectors $\vec{a} = 6\vec{\imath} - 2\vec{\jmath} - 3\vec{k}$ and $\vec{b} = \vec{\imath} + \vec{\jmath} + \vec{k}$.

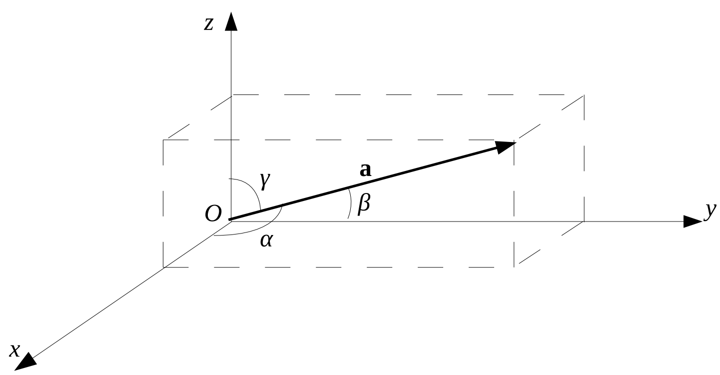
Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\pi/2$.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

Example 6. Determine whether the vectors $\vec{a} = 3\vec{\imath} + \vec{\jmath} - \vec{k}$ and $\vec{b} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$ are orthogonal, parallel or neither.

Example 7. Find the values of x such that the vectors $\vec{a} =< x, 1, 2 >$ and $\vec{b} =< 3, 4, x >$ are orthogonal.

Direction angles and direction cosines. The **direction angles** of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\vec{a}$.



$$\boxed{\cos\alpha = \frac{\vec{a}\cdot\vec{\imath}}{|\vec{a}||\vec{\imath}|} = \frac{a_1}{|\vec{a}|}}, \boxed{\cos\beta = \frac{a_2}{|\vec{a}|}}, \boxed{\cos\gamma = \frac{a_3}{|\vec{a}|}}.$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = 1$$

We can write

$$\vec{a} =< a_1, a_2, a_3 >=< |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma >=$$

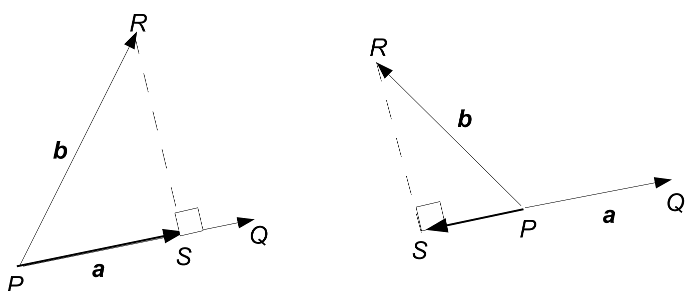
$$|\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} =< \cos\alpha, \cos\beta, \cos\gamma >$$

which says that the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$.

Example 8. Find the direction cosines of the vector $< -4, -1, 2 >$.



$\vec{PS} = \text{proj}_{\vec{a}}\vec{b}$ is called the **vector projection of $\vec{b}$ onto $\vec{a}$.**

$|\vec{PS}| = \text{comp}_{\vec{a}}\vec{b}$ is called the **scalar projection of $\vec{b}$ onto** $\vec{a}$ or the **component of $\vec{b}$ along** $\vec{a}$.

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \left|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right|} \quad \boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} < a_1, a_2, a_3 >}$$

Example 9. Find the scalar and vector projections of $\vec{b} =< 4, 2, 0 >$ onto $\vec{a} =< 1, 2, 3 >$.

Example 10. A constant force with vector representation $\vec{F} = 10\vec{\imath} + 18\vec{\jmath} - 6\vec{k}$ moves an object along a straight line from the point $(2, 3, 0)$ to the point $(4, 9, 15)$. Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

Example 11. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20^0 above the horizontal. How much work is done?