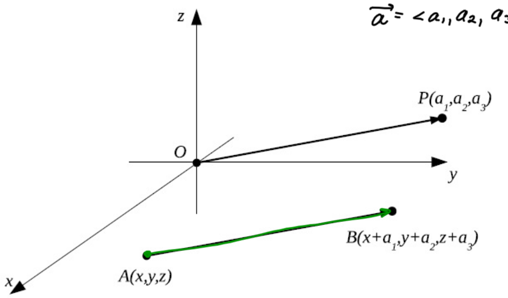
Section 11.2 **Vectors and the dot product in three dimensions**

Geometrically, a three-dimensional vector can be considered as an arrow with both a length and direction. An arrow is a directed line segment with a starting point and an ending point. Algebraically, a tree-dimensional vector is an ordered triple $\vec{a} =< a_1, a_2, a_3 >$ of real numbers. The numbers a_1, a_2, and a_3 are called the components of $\vec{a}$.

A representation of the vector $\vec{a} =< a_1, a_2, a_3 >$ is a directed line segment $\vec{AB}$ from any point $A(x, y, z)$ to the point $B(x + a_1, y + a_2, z + a_3)$.

A particular representation of $\vec{a} =< a_1, a_2, a_3 >$ is the directed line segment $\vec{OP}$ from the origin to the point $P(a_1, a_2, a_3)$, and $\vec{a} =< a_1, a_2, a_3 >$ is called the **position vector** of the point $P(a_1, a_2, a_3)$.

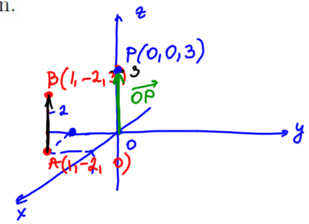
$\vec{a} = < a_1, a_2, a_3 >$



$\vec{OP}$ is the position vector of the point $P(a_1, a_2, a_3)$.

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then $\boxed{\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >}$.

Example 1. Find a vector $\vec{a}$ with representation given by the directed line segment $\vec{AB}$, where $A(1, -2, 0)$, $B(1, -2, 3)$. Draw $\vec{AB}$ and the equivalent representation starting at the origin.



Components of $\vec{AB}$:

$\vec{AB} = < 1-1, -2-(-2), 3-0 >$

$= < 0, 0, 3 >$

Position vector $\vec{OP}$, $P(0,0,3)$

The **magnitude (length)** $|\vec{a}|$ of $\vec{a}$ is the length of any its representation.

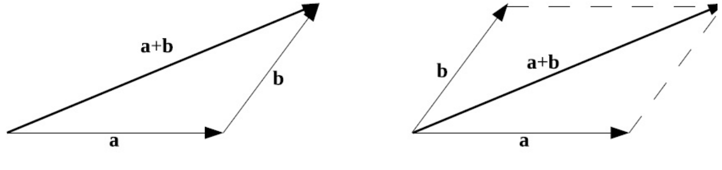
The length of $\vec{a}$ is $\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$

• $\vec{0} = <0,0,0>$

The only vector with length 0 is the **zero vector** $\vec{0} =< 0, 0, 0 >$. This vector is the only vector with no specific direction, its representation is a point.

Vector addition If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then the vector $\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$

$\vec{a} + \vec{b} = <a_1+b_1, a_2+b_2, a_3+b_3>$



Triangle Law

Parallelogram Law

$\vec{a}$ ↗ $2\vec{a}$ ↗ ↙ $-\frac{1}{2}\vec{a}$

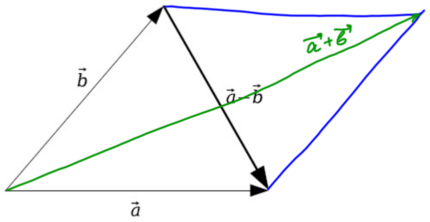
$c>0$ - same direction

$c<0$ - opposite direction.

Multiplication of a vector by a scalar If c is a scalar and $\vec{a} =< a_1, a_2, a_3 >$, then the vector $\boxed{c\vec{a} =< ca_1, ca_2, ca_3 >}$, $|c\vec{a}| = |c||\vec{a}|$

Two vectors $\vec{a}$ and $\vec{b}$ are called **parallel** if $\vec{b} = c\vec{a}$ for some scalar c. If $\vec{a} =< a_1, a_2, a_3 >$, $\vec{b} =< b_1, b_2, b_3 >$, then $\vec{a}$ and $\vec{b}$ are parallel if and only if $\boxed{\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}}$.

By the **difference** of two vectors, we mean $\boxed{\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >}$



Example 2. Find $|\vec{a}|$, $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, $2\vec{a} - 5\vec{b}$ if $\vec{a} =< 1, -3, 2 >$, $\vec{b} =< 2, 1, -1 >$.

$|2\vec{a} - 5\vec{b}|$ - ?

$2\vec{a} - 5\vec{b} = 2<1,-3,2> - 5<2,1,-1>$
$= <2-10, -6-5, 4+5>$

$\boxed{2\vec{a} - 5\vec{b} = <-8, -11, 9>}$

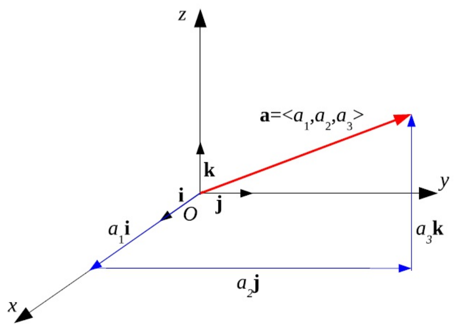
$|2\vec{a} - 5\vec{b}| = \sqrt{(-8)^2 + (-11)^2 + 9^2}$
$= \sqrt{64+121+81} = \sqrt{266}$

Properties of vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k and m are scalars, then

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
6. $(k + m)\vec{a} = k\vec{a} + m\vec{a}$
7. $(km)\vec{a} = k(m\vec{a})$
8. $1\vec{a} = \vec{a}$

Let $\vec{\imath} =< 1, 0, 0 >$ and $\vec{\jmath} =< 0, 1, 0 >$, $k =< 0, 0, 1 >$, $|\vec{\imath}| = |\vec{\jmath}| = |k| = 1$.

$$\boxed{\vec{a} =< a_1, a_2, a_3 >= a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}}$$



A **unit vector** is a vector whose length is 1.

A vector $\boxed{\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \right\rangle}$ is a unit vector that has the same direction as $\vec{a} =< a_1, a_2, a_3 >$.

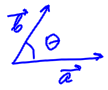
Example 3. Find the unit vector in the direction of the vector $\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$. $= \vec{a}$

$\vec{a} = <1, -2, 2>$

$|\vec{a}| = \sqrt{1 + (-2)^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{<1,-2,2>}{3} = \boxed{< \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} >}$

Definition. The **dot** or **scalar product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a}\cdot\vec{b} = 0$.



$0<\theta<\frac{\pi}{2}$

$\vec{a}\cdot\vec{b} > 0$ if and only if $0 < \theta < \pi/2$
$\vec{a}\cdot\vec{b} < 0$ if and only if $\pi/2 < \theta < \pi$

$\frac{\pi}{2}<\theta<\pi$

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then $\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$.

$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$$

Example 4. Given $\vec{a} =< 2, 3, -4 >$, $\vec{b} =< 1, -4, 8 >$. Find $\vec{a}\cdot\vec{b}$.

$$\vec{a}\cdot\vec{b} = 2(1) + (3)(-4) + (-4)(8) = \boxed{-42}$$

Example 5. Find the angle between vectors $\vec{a} = 6\vec{\imath} - 2\vec{\jmath} - 3\vec{k}$ and $\vec{b} = \vec{\imath} + \vec{\jmath} + \vec{k}$.

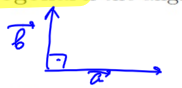
$\vec{a} = < 6, -2, -3 >$

$\vec{b} = < 1, 1, 1 >$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{6-2-3}{\sqrt{36+4+9}\sqrt{1+1+1}} = \boxed{\frac{1}{7\sqrt{3}}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{7\sqrt{3}}\right)}$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal if the angle between them is $\pi/2$.



$\cos\frac{\pi}{2} = 0$

$\boxed{\text{Two vectors } \vec{a} \text{ and } \vec{b} \text{ are orthogonal if and only if } \vec{a}\cdot\vec{b} = 0.}$

Example 6. Determine whether the vectors $\vec{a} = 3\vec{\imath} + \vec{\jmath} - \vec{k}$ and $\vec{b} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$ are orthogonal, parallel or neither.

$\vec{a} = <3, 1, -1>$

$\vec{b} = <1, -1, 2>$

$\vec{a}\cdot\vec{b} = 3 - 1 - 2 = 0$

$\boxed{\text{orthogonal}}$

$\frac{3}{1} \overset{?}{=} \frac{1}{-1} \overset{?}{=} -\frac{1}{2}$

not parallel

Example 7. Find the values of x such that the vectors $\vec{a} =< x, 1, 2 >$ and $\vec{b} =< 3, 4, x >$ are orthogonal.

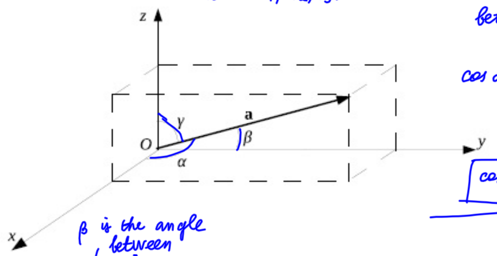
$0 = \vec{a}\cdot\vec{b} = 3x + 4 + 2x$, then solve for x

$5x = -4$

$\boxed{x = -\frac{5}{4}}$

Direction angles and direction cosines. The direction angles of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the direction cosines of the vector $\vec{a}$.

$\vec{a} = <a_1, a_2, a_3>$



α is the angle between $\vec{a}$ and $\vec{i} = <1,0,0>$

$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{<a_1,a_2,a_3>\cdot<1,0,0>}{|\vec{a}|}$

$\boxed{\cos\alpha = \frac{a_1}{|\vec{a}|}}$

β is the angle between $\vec{a} = <a_1, a_2, a_3>$ and $\vec{j} = <0,1,0>$.

γ is the angle between $\vec{a} = <a_1, a_2, a_3>$ and $\vec{k} = <0,0,1>$.

$$\boxed{\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|}}, \boxed{\cos\beta = \frac{a_2}{|\vec{a}|}}, \boxed{\cos\gamma = \frac{a_3}{|\vec{a}|}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = 1$$

We can write

$$\vec{a} =< a_1, a_2, a_3 >=< |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma >=$$

$$\boxed{\vec{a} = <a_1, a_2, a_3> = |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >}$$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} =< \cos\alpha, \cos\beta, \cos\gamma >$$

which says that the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$.

Example 8. Find the direction cosines of the vector $< -4, -1, 2 >$. = Find a unit vector in the direction of the vector $<-4,-1,2>$

$\vec{a} = <-4,-1,2>$

$|\vec{a}| = \sqrt{(-4)^2 + (-1)^2 + 2^2} = \sqrt{16+1+4} = \sqrt{21}$

$$\frac{\vec{a}}{|\vec{a}|} = \frac{<-4,-1,2>}{\sqrt{21}} = <\underbrace{-\frac{4}{\sqrt{21}}}_{\cos\alpha}, \underbrace{-\frac{1}{\sqrt{21}}}_{\cos\beta}, \underbrace{\frac{2}{\sqrt{21}}}_{\cos\gamma}>$$

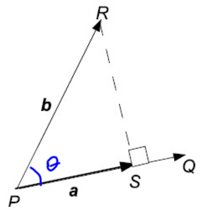
$$\boxed{\cos\alpha = -\frac{4}{\sqrt{21}}, \quad \cos\beta = -\frac{1}{\sqrt{21}}, \quad \cos\gamma = \frac{2}{\sqrt{21}}}$$

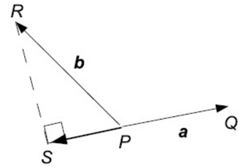
Title : Apr 22-10:55 AM (Page 8 of 10)

$|\vec{PS}| = |\vec{PR}|\cos\theta$

$= |\cancel{\vec b}| \cdot \frac{\vec a \cdot \vec b}{|\vec a| \cdot |\cancel{\vec b}|}$

$\boxed{comp_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|}}$





If $\frac{\pi}{2} < \theta < \pi$, then $\vec a \cdot \vec b < 0$ and $comp_{\vec a}\vec b < 0$.

if $0 < \theta < \frac{\pi}{2}$

$\vec{PS} = \text{proj}_{\vec a}\vec b$ is called the **vector projection of $\vec b$ onto $\vec a$**.

$|\vec{PS}| = \text{comp}_{\vec a}\vec b$ is called the **scalar projection of $\vec b$ onto $\vec a$** or the **component of $\vec b$ along** $\vec a$.

$\boxed{\text{comp}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|}}$ $\boxed{\text{proj}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|^2}\vec a = \frac{\vec a \cdot \vec b}{|\vec a|^2} < a_1, a_2, a_3 >}$ $\leftarrow$ $proj_{\vec a}\vec b = (comp_{\vec a}\vec b) \cdot \frac{\vec a}{|\vec a|}$

Example 9. Find the scalar and vector projections of $\vec b = < 4, 2, 0 >$ onto $\vec a = < 1, 2, 3 >$.

$comp_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|} = \frac{<1,2,3> \cdot <4,2,0>}{\sqrt{1+4+9}} = \frac{4+4}{\sqrt{14}} = \boxed{\frac{8}{\sqrt{14}}}$

$proj_{\vec a}\vec b = (comp_{\vec a}\vec b) \cdot \frac{\vec a}{|\vec a|} = \frac{8}{\sqrt{14}} \cdot \frac{<1,2,3>}{\sqrt{14}} = \frac{1}{14} < 8, 16, 24 >$

$= \boxed{\frac{1}{7} < 4, 8, 12 >}$

Example 10. A constant force with vector representation $\vec{F} = 10\vec{\imath} + 18\vec{\jmath} - 6\vec{k}$ moves an object along a straight line from the point $(2, 3, 0)$ to the point $(4, 9, 15)$. Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

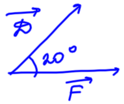
$$W = \vec{F} \cdot \vec{d}$$

$$\vec{d} = \langle 4-2, 9-3, 15-0 \rangle = \langle 2, 6, 15 \rangle$$

$$W = \langle 10, 18, -6 \rangle \cdot \langle 2, 6, 15 \rangle$$

$$= 20 + 108 - 90 = \boxed{38}\ (J)$$

Example 11. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20^0 above the horizontal. How much work is done?



$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos 20°$$

$$|\vec{F}| = 25$$

$$|\vec{D}| = 10$$

$$= \boxed{25(10) \cos(20°)}$$

$$\approx \boxed{234.92\ (lb\text{-}ft)}$$