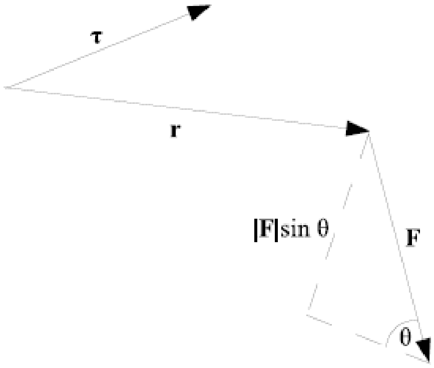
Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.3 **The cross product.**

If we tighten a bolt by applying a force to a wrench, we produce a turning effect called a *torque* $\vec{\tau}$ that acts along the axis of the bolt to move it forward.



The magnitude of the torque depends on two things:

- The distance from the axis of the bolt to the point where the force is applied. This is $|\vec{r}|$, the length of the position vector $\vec{r}$.
- The scalar component of the force $\vec{F}$ in the direction perpendicular to $\vec{r}$. This is the only component that can cause a rotation and it is

$$|\vec{F}|\sin\theta$$

where θ is an angle between the vectors $\vec{r}$ and $\vec{F}$.

We define the magnitude of the torque to be the product of these two factors:

$$|\tau| = |\vec{r}||\vec{F}|\sin\theta$$

If $\vec{n}$ is a unit vector that points in the direction in which a right-threaded bolt moves, we define the torque to be the vector

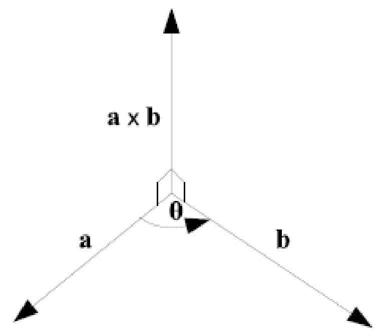
$$\vec{\tau} = (|\vec{r}||\vec{F}|\sin\theta)\vec{n}.$$

We denote this torque vector by $\vec{\tau} = \vec{r} \times \vec{F}$ and we call it the *cross product* or *vector product* of $\vec{r}$ and $\vec{F}$.

Definition. If $\vec{a}$ and $\vec{b}$ are nonzero three-dimensional vectors, the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

$$\boxed{\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\vec{n}}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and whose direction is given by the right-hand rule: If the fingers of your hand curl through the angle θ from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{n}$.



If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, then we define $\vec{a} \times \vec{b}$ to be $\vec{0}$. $\uparrow\vec{a}$ $\uparrow\vec{b}$ $\theta = 0$ $\sin\theta = 0$

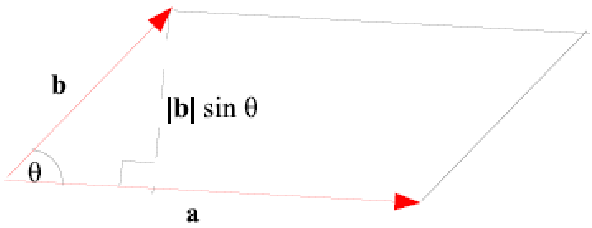
$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

$\boxed{\text{Two nonzero vectors } \vec{a} \text{ and } \vec{b} \text{ are parallel if and only if } \vec{a} \times \vec{b} = \vec{0}.}$ $\uparrow\vec{a}$ $\downarrow\vec{b}$ $\theta = \pi$ $\sin\theta = 0$.

Properties of the cross product. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



Volume = $|\vec{a} \times \vec{b}|$

magnitude of the vector

The cross product in component form.
A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} \boxed{a_1} & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & \boxed{a_2} & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 & \boxed{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The cross product of a $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ is

$$\boxed{\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}}$$

Example 1. If $\vec{a} =< -2, 3, 4 >$ and $\vec{b} =< 3, 0, 1 >$, find $\vec{a} \times \vec{b}$.

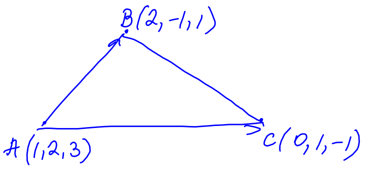
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} = \vec{\imath}\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix}$$

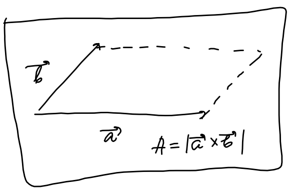
$$\begin{vmatrix} \boxed{\vec{\imath}} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} \quad \begin{vmatrix} \vec{\imath} & \boxed{\vec{\jmath}} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} \quad \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \boxed{\vec{k}} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \vec{\imath}\left((3)(1) - (0)(4)\right) - \vec{\jmath}\left((-2)(1) - (3)(4)\right) + \vec{k}\left((-2)(0) - (3)(3)\right)$$

$$= 3\vec{\imath} + 14\vec{\jmath} - 9\vec{k} = \boxed{< 3, 14, -9 >}$$

Example 2. Find the area of the triangle with vertices $A(1, 2, 3)$, $B(2, -1, 1)$, $C(0, 1, -1)$.





$$A = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$\vec{AB} = < 2-1, -1-2, 1-3 > = < 1, -3, -2 >$$

$$\vec{AC} = < 0-1, 1-2, -1-3 > = < -1, -1, -4 >$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -3 & -2 \\ -1 & -1 & -4 \end{vmatrix}$$

$$= \vec{\imath}\left((-3)(-4) - (-1)(-2)\right) - \vec{\jmath}\left((1)(-4) - (-1)(-2)\right) + \vec{k}\left((1)(-1) - (-1)(-3)\right)$$

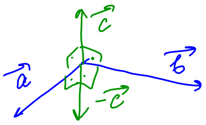
$$= 10\vec{\imath} + 6\vec{\jmath} - 4\vec{k}$$

$$A = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{10^2 + (6)^2 + (-4)^2}$$

$$= \boxed{\frac{1}{2}\sqrt{152}} = \frac{1}{2}\sqrt{38 \cdot 4} = \boxed{\sqrt{38}}$$

Example 3. Find two unit vectors orthogonal to both $\vec{\imath}+\vec{\jmath}$ and $\vec{\imath}-\vec{\jmath}+\vec{k}$.

$|\vec{c}|=1$



$\vec{a} = \vec{\imath} + \vec{\jmath} = \langle 1, 1, 0 \rangle$

$\vec{b} = \vec{\imath} - \vec{\jmath} + \vec{k} = \langle 1, -1, 1 \rangle$

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \vec{\imath}\left(1-(-1)(0)\right) - \vec{\jmath}\left(1-(1)(0)\right) + \vec{k}\left(1(-1)-1\right)$$

$$= \vec{\imath} - \vec{\jmath} - 2\vec{k} = \langle 1, -1, -2 \rangle$$

$$\vec{c} = \frac{\langle 1,-1,-2 \rangle}{|\langle 1,-1,-2\rangle|} = \frac{\langle 1,-1,-2\rangle}{\sqrt{1+1+4}} = \boxed{\frac{1}{\sqrt{6}}\langle 1,-1,-2\rangle}$$

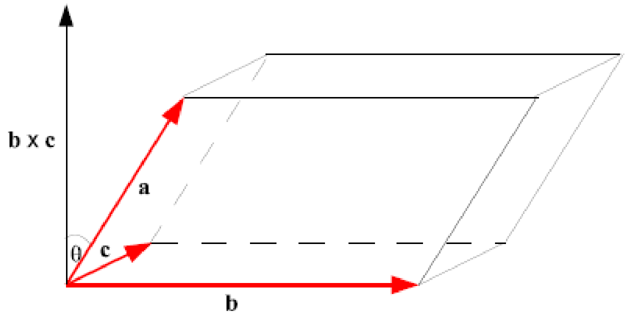
$$\boxed{-\vec{c} = -\frac{1}{\sqrt{6}}\langle 1,-1,-2\rangle}$$

Triple products

The product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|.$$



$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given in component form:

$$\vec{a} =< a_1, a_2, a_3 >,\quad \vec{b} =< b_1, b_2, b_3 >,\quad \vec{c} =< c_1, c_2, c_3 > .$$

Then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \left|\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right| \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}$$

$$= \boxed{a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1}$$

Example 4. Find the volume of the parallelepiped determined by vectors $\vec{a} = 2\vec{\imath}+3\vec{\jmath}-2\vec{k}$, $\vec{b} = \vec{\imath}-\vec{\jmath}$, and $\vec{c} = 2\vec{\imath}+3\vec{k}$.

$\vec{a} = < 2, 3, -2 >$

$\vec{b} = < 1, -1, 0 >$

$\vec{c} = < 2, 0, 3 >$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \left|\begin{array}{ccc} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{array}\right| \begin{array}{ccc} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{array}$$

$$= (2)(-1)(3) + (3)(0)(2) + (-2)(1)(0) - (-2)(-1)(2) - (3)(1)(3) - 2(0)(0)$$

$$= -6 - 4 - 9 = -19$$

$$V = |-19| = \boxed{19}$$

Example 5. Use the scalar triple product to verify that the vectors $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$, $\vec{b} = \vec{\imath} - \vec{\jmath}$, and $\vec{c} = 7\vec{\imath} + 3\vec{\imath} + 2\vec{k}$ are complanar; that is, they lie in the same plane.

$\boxed{\vec{a}, \vec{b}, \vec{c} \text{ are complanar if and only if } \vec{a}\cdot(\vec{b}\times\vec{c}) = 0}$

$\vec{a} = \langle 2, 3, 1 \rangle$

$\vec{b} = \langle 1, -1, 0 \rangle$

$\vec{c} = \langle 7, 3, 2 \rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} \begin{matrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{matrix} = (2)(-1)(2) + (3)(0)(7) + (1)(1)(3) - (1)(-1)(7) - 2(1)(3) - 3(0)(2)$$

$$= -4 + 3 + 7 - 6 = \boxed{0}$$

$\boxed{\text{complanar}}$

The product $\vec{a} \times (\vec{b} \times \vec{c})$ is called the vector triple product of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.$$