Section 6.5 The Substitution Rule

The substitution rule for indefinite integrals. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du}$$

substitution $u = g(x)$
differential of u is $du = g'(x)dx$

Example 1. Evaluate each integral:

1. $\frac{1}{5}\int 5\sin 5x\,dx$

$u = 5x$, $du = (5x)'dx$, $du = 5dx$

$= \frac{1}{5}\int \sin u\,du$
$= -\frac{1}{5}\cos u + C$
$= \boxed{-\frac{1}{5}\cos(5x) + C}$

2. $\frac{1}{3}\int \frac{3\,dx}{\sqrt{3x+1}}$

$u = 3x+1$, $du = (3x+1)'dx$, $du = 3\,dx$

$= \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{-1/2+1}}{-1/2+1} + C$
$= \frac{1}{3}\frac{u^{1/2}}{1/2} + C = \frac{2}{3}u^{1/2} + C$
$= \boxed{\frac{2}{3}(3x+1)^{1/2} + C}$

3. $\frac{1}{3}\int 3x^2 e^{x^3}\,dx$

$u = x^3$, $du = 3x^2dx$

$= \frac{1}{3}\int e^u du$
$= \frac{1}{3}e^u + C$
$= \boxed{\frac{1}{3}e^{x^3} + C}$

4. $\int \frac{2x^2+4x}{x^3+3x^2-4}dx$

$u = x^3+3x^2-4$
$du = (x^3+3x^2-4)'dx$
$du = (3x^2+6x)dx$
$du = 3(x^2+2x)dx$

$= \frac{1}{3}\int \frac{3 \cdot 2(x^2+2x)dx}{x^3+3x^2-4}$

$= \frac{2}{3}\int \frac{\overbrace{3(x^2+2x)dx}^{=du}}{\underbrace{x^3+3x^2-4}_{=u}} = \frac{2}{3}\int \frac{du}{u} = \frac{2}{3}\ln|u| + C$

$= \boxed{\frac{2}{3}\ln|x^3+3x^2-4| + C}$

$\int x^{-1/2}dx = 2x^{1/2}+C$

5. $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx \quad = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx + \int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

$v = 1-x^2$

$dv = -2x\,dx$

$= -\frac{1}{2}\int \frac{dv}{\sqrt{v}}$

$= -\frac{1}{2}\int v^{-1/2}\,dv$

$= -\frac{1}{\cancel{2}}(\cancel{2})v^{1/2}+C$

$= -v^{1/2}+C$

$= -(1-x^2)^{1/2}+C$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}}dx$

$= \int u\,du = \frac{u^2}{2}+C$

$= \frac{(\arcsin x)^2}{2}+C$

$$\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx = \boxed{-(1-x^2)^{1/2} + \frac{(\arcsin x)^2}{2} + C}$$

<ins>**The substitution rule for definite integrals.**</ins> If $g'(x)$ is continuous on $[a,b]$ and f is continuous on the range of g, then

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du}$$

Example 2. Evaluate the integral:

1. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$u = \ln x$
$du = \frac{1}{x}dx$
$u = \ln x$
$e \to \ln e = 1$
$e^4 \to \ln(e^4) = 4$

$$= \int_1^4 \frac{du}{\sqrt{u}} = \int_1^4 u^{-1/2}du = 2u^{1/2}\Big]_1^4$$

$$= 2(4^{1/2} - 1^{1/2}) = \boxed{2}$$

2. $\displaystyle\frac{1}{2}\int_0^1 \frac{2x\,dx}{\sqrt{1+x^4}}$

$\cancel{u = 1+x^4}$
$\cancel{du = 4x^3dx}$

$u = x^2$
$du = 2x\,dx$
$0 \to 0^2 = 0$
$1 \to 1^2 = 1$

$$= \frac{1}{2}\int_0^1 \frac{du}{\sqrt{1+u^2}}$$

$$= \frac{1}{2}\ln\left|u + \sqrt{1+u^2}\right|\Big]_0^1$$

$$= \frac{1}{2}\left(\ln\left|1+\sqrt{1+1}\right| - \ln\left|0+\sqrt{1+0}\right|\right)$$

$$= \frac{1}{2}\left(\ln|1+\sqrt{2}| - \cancelto{0}{\ln 1}\right)$$

3. $\displaystyle\int_{-3/2}^{0} \frac{x\,dx}{\sqrt{1-2x}}$

$u = 1-2x \longrightarrow x = \frac{1-u}{2}$

$dx = \left(\frac{1-u}{2}\right)' du$

$dx = -\frac{1}{2} du$

$-\frac{3}{2} \longmapsto 1-2\left(-\frac{3}{2}\right) = 4$

$0 \longmapsto 1-2(0) = 1$

$$= \int_4^1 \frac{\frac{1-u}{2}\left(-\frac{1}{2}\right) du}{\sqrt{u}}$$

$$= -\frac{1}{4}\int_4^1 \frac{(1-u)}{\sqrt{u}}\, du$$

$$= \frac{1}{4}\int_1^4 (1-u)\,u^{-1/2}\, du$$

$$= \frac{1}{4}\int_1^4 \left(u^{-1/2} - u^{1/2}\right) du$$

$$= \frac{1}{4}\left[\frac{u^{-1/2+1}}{-1/2+1} - \frac{u^{1/2+1}}{u^{1/2+1}}\right]_1^4$$

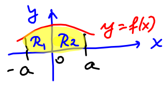
$$= \frac{1}{4}\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]_1^4$$

$$= \frac{1}{4}\left(2(4^{1/2}) - \frac{2}{3}(4^{3/2}) - 2 + \frac{2}{3}\right) = \boxed{\frac{1}{4}\left(2(2) - \frac{2}{3}(8) - 2 + \frac{2}{3}\right)}$$

Integrals of symmetric functions. Suppose f is continuous on $[-a, a]$.

- If f is **even**, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

$f(-x) = f(x)$

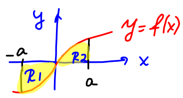


$A(R_1) = A(R_2) = \int_0^a f(x)dx$

$\int_{-a}^{a} f(x)dx = A(R_1) + A(R_2) = 2A(R_2) = 2\int_0^a f(x)dx$

- If f is **odd**, then $\int_{-a}^{a} f(x)dx = 0$

$f(-x) = -f(x)$



$A(R_1) = A(R_2)$

$\int_{-a}^{a} f(x)dx = A(R_2) - A(R_1) = 0$

Example 3. Evaluate the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx. = \boxed{0}$

x^2 is even

$1+x^6$ is even

$\frac{x^2}{1+x^6} = \frac{\text{even}}{\text{even}} = \text{even}$

$\sin x$ is odd

$\frac{x^2}{1+x^6}\sin x = (\text{even})(\text{odd}) = \text{odd}$