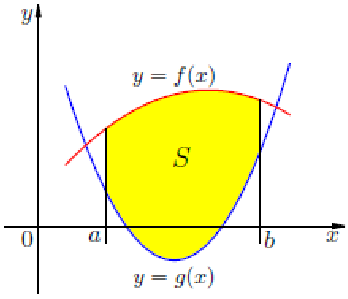
Chapter 7. **Applications of integration**
Section 7.1 **Areas between curves**

The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$, is

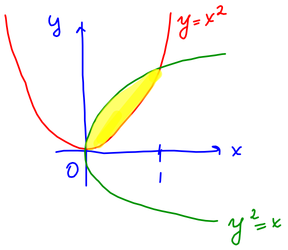
$$\boxed{A = \int_a^b \overset{[\text{top}] - [\text{bottom}]}{[f(x) - g(x)]}\,dx}$$



Example 1. Find the area of the region bounded by

1. $y = x^2$, $y^2 = x$

$y = \sqrt{x}$



Points of intersection:

$(x^2)^2 = (\sqrt{x})^2$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$\boxed{x=0}$ or $x^3 - 1 = 0$

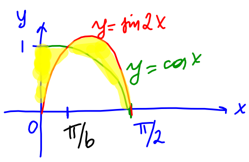
$x^3 = 1$

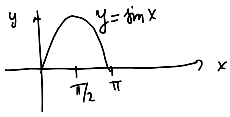
$\boxed{x=1}$

$$\text{area} = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left[\frac{x^{1/2+1}}{1/2+1} - \frac{x^3}{3}\right]_0^1 = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

2. $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$





Points of intersection:

$\cos x = \sin 2x$ $\qquad \sin 2x = 2 \sin x \cos x$

$\cos x = 2 \cos x \sin x$

$\cos x - 2 \cos x \sin x = 0$

$\cos x (1 - 2 \sin x) = 0$

$\cos x = 0$ or $1 - 2 \sin x = 0$

$\boxed{x = \frac{\pi}{2}}$ $\qquad \sin x = \frac{1}{2}$

$\boxed{x = \frac{\pi}{6}}$

$$\text{area} = \int_0^{\pi/6} (\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\,dx$$

$$= \int_0^{\pi/6} \cos x\,dx - \frac{1}{2}\int_0^{\pi/6} 2 \sin 2x\,dx + \frac{1}{2}\int_{\pi/6}^{\pi/2} 2 \sin 2x\,dx - \int_{\pi/6}^{\pi/2} \cos x\,dx$$

$u = 2x$

$du = 2\,dx$

$0 \to 2(0) = 0$

$\frac{\pi}{6} \to 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$

$\frac{\pi}{2} \to 2\left(\frac{\pi}{2}\right) = \pi$

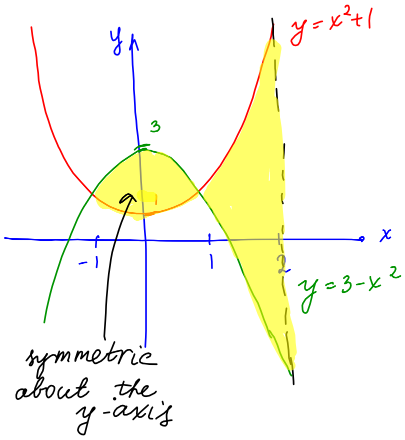
$$= \int_0^{\pi/6} \cos x\,dx - \frac{1}{2}\int_0^{\pi/3} \sin u\,du + \frac{1}{2}\int_{\pi/3}^{\pi} \sin u\,du - \int_{\pi/6}^{\pi/2} \cos x\,dx$$

$$= \sin x \Big]_0^{\pi/6} + \frac{1}{2} \cos u \Big]_0^{\pi/3} - \frac{1}{2} \cos u \Big]_{\pi/3}^{\pi} - \sin x \Big]_{\pi/6}^{\pi/2}$$

$$= \overset{1/2}{\sin \frac{\pi}{6}} - \overset{0}{\sin 0} + \frac{1}{2} \overset{1/2}{\cos \frac{\pi}{3}} - \frac{1}{2} \overset{1}{\cos 0} - \frac{1}{2} \overset{-1}{\cos \pi} + \frac{1}{2} \overset{1/2}{\cos \frac{\pi}{3}} - \overset{1}{\sin \frac{\pi}{2}} + \overset{1/2}{\sin \frac{\pi}{6}}$$

$$= \boxed{\frac{1}{2} + \frac{1}{4} - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} - 1 + \frac{1}{2}}$$

3. $y = x^2 + 1$, $y = 3 - x^2$, $x = -1$, $x = 2$



Points of intersection:

$$x^2 + 1 = 3 - x^2$$
$$2x^2 = 2$$
$$x^2 = 1$$
$$x = \pm 1$$

$$\text{area} = \int_{-1}^{1} \left(3 - x^2 - (x^2+1)\right) dx + \int_{1}^{2} \left(x^2 + 1 - (3 - x^2)\right) dx$$

$$= 2\int_{0}^{1} (2 - 2x^2)\, dx + \int_{1}^{2} (2x^2 - 2)\, dx$$

$$= 2\left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2$$

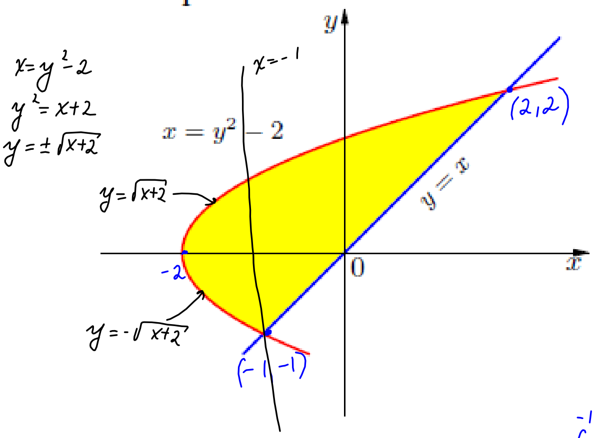
$$= 2\left[2 - \frac{2}{3}\right] + \left[\frac{2(2^3)}{3} - 2(2) - \frac{2}{3} + 2\right]$$

$$= \boxed{2\left(2 - \frac{2}{3}\right) + \left(\frac{16}{3} - 4 - \frac{2}{3} + 2\right)}$$

In general case, the area between the curves $y = f(x)$, $y = g(x)$ and between $x = a$ and $x = b$, is

$$\boxed{A = \int_a^b |f(x) - g(x)|dx}$$

Example 2. Find the area of the shaded region.



Points of intersection:

$y^2 - 2 = y$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y_1 = -1, \quad y_2 = 2$

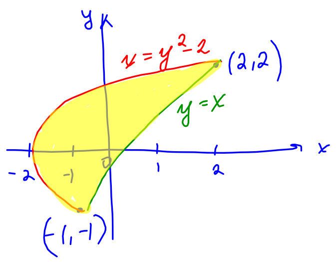
Both points lie on the line $y = x$

corresponding x-coordinate.

$x_1 = -1, \quad x_2 = 2$

Points of intersection: $(-1,-1)$ and $(2,2)$

$$\text{area} = \int_{-2}^{-1} \left(\sqrt{x+2} - (-\sqrt{x+2})\right)dx + \int_{-1}^{2} \left(\sqrt{x+2} - x\right)dx$$



Integrate for y.

$$\boxed{\text{area} = \int_a^b \left([\text{right}] - [\text{left}]\right)dy}$$

$-1 \le y \le 2$

$$\text{area} = \int_{-1}^{2} \left(y - (y^2 - 2)\right)dy$$

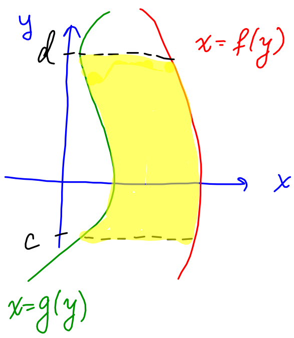
$$= \int_{-1}^{2} \left(y - y^2 + 2\right)dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} + 2y\right]_{-1}^{2}$$

$$= \frac{2^2}{2} - \frac{2^3}{3} + 2(2) - \left(\frac{(-1)^2}{2} - \frac{(-1)^3}{3} + 2(-1)\right)$$

$$= \boxed{2 - \frac{8}{3} + 4 - \left(\frac{1}{2} + \frac{1}{3} - 2\right)}$$

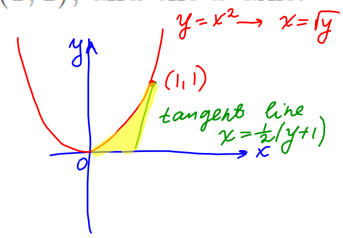
If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous functions and $f(y) \geq g(y)$ for all y in $[c, d]$, then its area is



$$\boxed{A = \int_c^d \underbrace{[f(y) - g(y)]}_{\text{[right]}-\text{[left]}} dy}$$

Example 3.

Find the area of the region bounded by the parabola $y = x^2$, tangent line to this parabola at $(1, 1)$, and the x-axis.



1. Equation of the tangent line.

$y-\cancel{f(1)}^{\,1} = \cancel{f'(1)}^{\,2}(x-1)$, where

$f(x)=x^2$

$f(1)=1$

$f'(x)=2x$

$f'(1)=2$

$y-1=2(x-1)$ tangent line

tangent line:

$y-1=2x-2$

$2x=y+1$

$x=\frac{1}{2}(y+1)$

2. Integrate for y.

$0\le y\le 1$

$$\text{area} = \int_0^1 \left(\frac{1}{2}(y+1)-\sqrt{y}\right)dy$$

$$= \left[\frac{1}{2}\left(\frac{y^2}{2}+y\right)-\frac{y^{3/2}}{3/2}\right]_0^1$$

$$= \left[\frac{1}{2}\left(\frac{y^2}{2}+y\right)-\frac{2}{3}y^{3/2}\right]_0^1$$

$$= \boxed{\frac{1}{4}+\frac{1}{2}-\frac{2}{3}}$$