## Section 7.2 **Volume**

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region $B_1$, called the **base**, and a congruent region $B_2$ in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join $B_1$ and $B_2$. If the area of the base is $A$ and the height of the cylinder is $h$, then the volume of the cylinder is defined as $V = Ah$.

Let $S$ be any solid. The intersection of $S$ with a plane is a plane region that is called a **cross-section** of $S$. Suppose that the area of the cross-section of $S$ in a plane $P_x$ perpendicular to the $x$-axis and passing through the point $x$ is $A(x)$, where $a \leq x \leq b$.

Let's consider a partition $P$ of $[a, b]$ by points $x_i$ such that $a = x_0 < x_1 < ... < x_n = b$. The planes $P_{x_i}$ will slice $S$ into smaller "slabs". If we choose $x_i^*$ in $[x_{i-1}, x_i]$, we can approximate the $i$th slab $S_i$ (the part of $S$ between $P_{x_{i-1}}$ and $P_{x_i}$) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the $i$th slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of $S$ is $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

**Definition of volume** Let $S$ be a solid that lies between the planes $P_a$ and $P_b$. If the cross-sectional area of $S$ in the plane $P_x$ is $A(x)$, where $A$ is an integrable function, then the **volume** of $S$ is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

IMPORTANT. $A(x)$ is the area of a moving cross-sectional obtained by slicing through $x$ perpendicular to the $x$-axis.

**Example 1.** Find the volume of a right circular cone with height $h$ and base radius $r$.

**Example 2.** Find the volume of a frustum of a pyramid with square base of side $b$, square top of side $a$, and height $h$.

**Volume by disks.** Let $S$ be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the $x$-axis.

A cross-section through $x$ perpendicular to the $x$-axis is a circular disc with radius $|y| = |f(x)|$, the cross-sectional area is $A(x) = \pi y^2 = \pi[f(x)]^2$, thus, we have the following **formula for a volume of revolution:**

$$V_X = \pi \int\limits_a^b [f(x)]^2 dx$$

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the $y$-axis.

Then the corresponding volume of revolution is

$$V_Y = \pi \int\limits_c^d [g(y)]^2 dy$$

**Volume by washers.** Let $S$ be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all $x$ in $[a, b]$ ) is rotated about the $x$-axis.

Then the volume of $S$ is

$$V_X = \pi \int\limits_a^b \{[f(x)]^2 - [g(x)]^2\} dx$$

**Example 3.**

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 2y$ about the $x$-axis.

2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the $y$-axis.

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about the line $y = 2$.