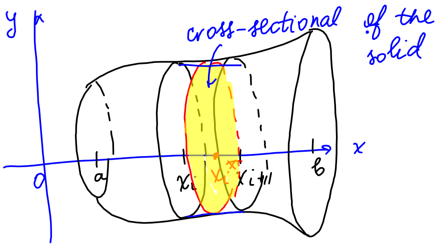
Section 7.2 **Volume**

We start with a simple type of solid called a **cylinder**. A cylinder is bounded by a plane region B_1, called the **base**, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments perpendicular to the base that join B_1 and B_2. If the area of the base is A and the height of the cylinder is h, then the volume of the cylinder is defined as $V = Ah$.

Let S be any solid. The intersection of S with a plane is a plane region that is called a **cross-section** of S. Suppose that the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x is $A(x)$, where $a \le x \le b$.



Partition $[a, b]$ into n subintervals.
"slice" the cylinder by the planes $x = x_i$, $i = 0, 1, \dots, n$
Take a "slice" of the solid between $x = x_i$ and x_{i+1}, approximate it by the cylinder of height $\Delta x_i = x_{i+1} - x_i$ and base area $A(x_i^*)$ $(x_i \le x_i^* \le x_{i+1})$
$A(x_i^*)$ is the area of the cross-sectional.
The volume of the cylinder is $V_i = A(x_i^*)\Delta x_i$

Let's consider a partition P of $[a, b]$ by points x_i such that $a = x_0 < x_1 < ... < x_n = b$. The planes P_{x_i} will slice S into smaller "slabs". If we choose x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S between $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and height $\Delta x_i = x_i - x_{i-1}$.

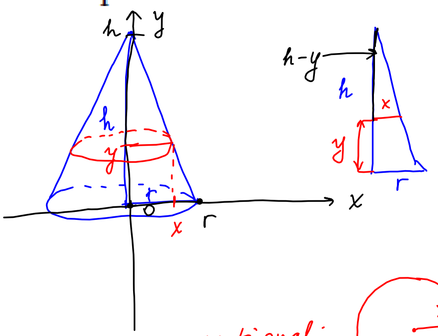
The volume of this cylinder is $A(x_i^*)\Delta x_i$, so the approximation to volume of the ith slab is $V(S_i) \approx A(x_i^*)\Delta x_i$. Thus, the approximation to the volume of S is $\boxed{V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i.}$ This approximation appears to become better and better as $\|P\| \to 0$.

Definition of volume Let S be a solid that lies between the planes P_a and P_b. If the cross-sectional area of S in the plane P_x is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \boxed{\int_a^b A(x)dx = V}$$

IMPORTANT. $A(x)$ is the area of a moving cross-sectional obtained by slicing through x perpendicular to the x-axis.

Example 1. Find the volume of a right circular cone with height h and base radius r.



Integrate for y.

$0 \le y \le h$

area of the cross-sectional $= \pi x^2$

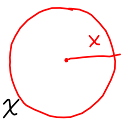
similar triangles:

$\frac{x}{r} = \frac{h-y}{h}$ — solve for x

$x = \frac{r}{h}(h-y)$

$A(y) = \pi x^2 = \pi\left[\frac{r}{h}(h-y)\right]^2$

cross-sectional: disk of radius x



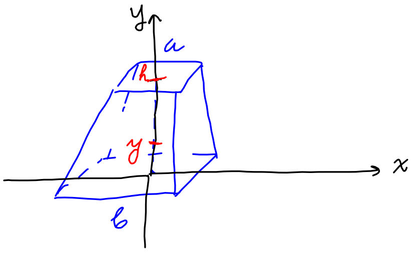
$A(y) = \pi \frac{r^2}{h^2}\left(h^2 - 2hy + y^2\right)$

$$V = \int_0^h A(y)\,dy = \pi \frac{r^2}{h^2} \int_0^h \left(h^2 - 2hy + y^2\right) dy$$

$$= \pi \frac{r^2}{h^2}\left[h^2 y - \cancel{2}h \frac{y^2}{\cancel{2}} + \frac{y^3}{3}\right]_0^h$$

$$= \pi \frac{r^2}{h^2}\left(\cancel{h^3} - \cancel{h^3} + \frac{h^3}{3}\right) = \frac{\pi r^2 h^3}{3h^2} = \boxed{\frac{1}{3}\pi r^2 h}$$

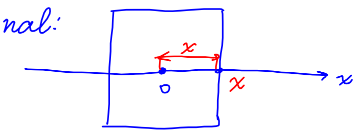
Example 2. Find the volume of a frustum of a pyramid with square base of side b, square top of side a, and height h.

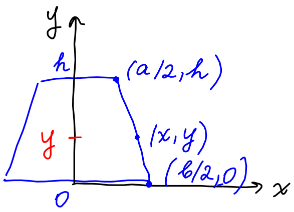


Pick $0 \le y \le h$

Slice the pyramid by the plane perpendicular to the y-axis through y.

Cross-sectional:





$$\text{area} = (2x)^2 = 4x^2$$

Find the relation between x and y.

Equation of the line through $\left(\frac{a}{2}, h\right)$ and $\left(\frac{b}{2}, 0\right)$

$$y - 0 = \frac{h-0}{\frac{a}{2} - \frac{b}{2}}\left(x - \frac{b}{2}\right)$$

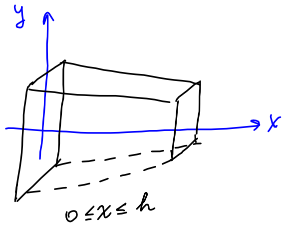
$$\frac{y}{\frac{2h}{a-b}} = \frac{\frac{2h}{a-b}\left(x - \frac{b}{2}\right)}{\frac{2h}{a-b}}$$

solve for x:

$$x - \frac{b}{2} = \frac{y(a-b)}{2h}$$

$$x = \frac{b}{2} + y\,\frac{a-b}{2h}$$

$$2x = b + y\,\frac{a-b}{h}$$



$$A(y) = \left(b + y\,\frac{a-b}{h}\right)^2 = b^2 + 2by\,\frac{a-b}{h} + y^2\left(\frac{a-b}{h}\right)^2$$

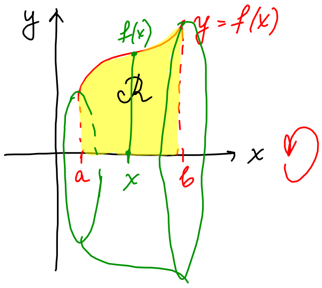
$$V = \int_0^h A(y)\,dy = \int_0^h \left[b^2 + 2b\boxed{y}\,\frac{a-b}{h} + \boxed{y^2}\left(\frac{a-b}{h}\right)^2\right]dy$$

$$= \left[b^2 y + \cancel{2}b\,\frac{y^2}{\cancel{2}}\,\frac{a-b}{h} + \frac{y^3}{3}\,\frac{(a-b)^2}{h^2}\right]_0^h$$

$$= b^2 h + b\,h^{\cancel{2}}\,\frac{a-b}{\cancel{h}} + \frac{h^{\cancel{3}}}{3}\,\frac{(a-b)^2}{\cancel{h^2}}$$

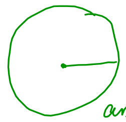
$$= \boxed{b^2 h + bh(a-b) + \frac{h}{3}(a-b)^2}$$

Volume by disks. Let S be the solid obtained by revolving the plane region $\mathcal{R}$ bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ about the x-axis.



$a \le x \le b$

cross-sectional through x



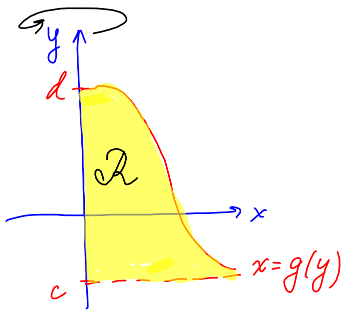
radius = $f(x)$

area = $A(x) = \pi r^2 = \pi[f(x)]^2$

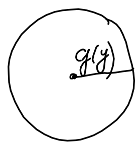
A cross-section through x perpendicular to the x-axis is a circular disc with radius $|y| = |f(x)|$, the cross-sectional area is $A(x) = \pi y^2 = \pi[f(x)]^2$, thus, we have the following **formula for a volume of revolution:**

$$\boxed{V_X = \pi \int_a^b [f(x)]^2 dx}$$

The region bounded by the curves $x = g(y)$, $x = 0$, $y = c$, and $y = d$ is rotated about the y-axis.



cross-sectional through $c \le y \le d$ is a circle of radius $g(y)$



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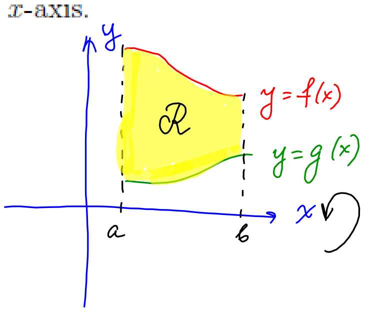
Then the corresponding volume of revolution is

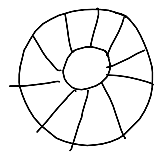
$$\boxed{V_Y = \pi \int_c^d [g(y)]^2 dy}$$

c

Volume by washers. Let S be the solid generated when the region bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ (where $f(x) \geq g(x)$ for all x in $[a, b]$) is rotated about the x-axis.



cross-sectional through $a \leq x \leq b$ is a washer with



inner radius $IR = g(x)$

outer radius $OR = f(x)$

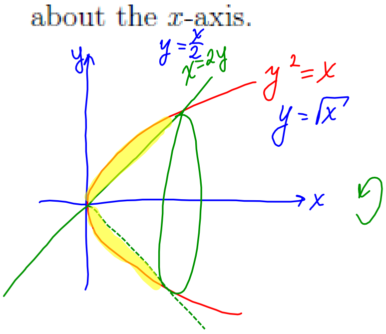
$A(x) = \pi([f(x)]^2 - [g(x)]^2)$

Then the volume of S is

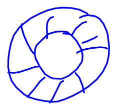
$$\boxed{V_X = \pi \int_a^b \{\overbrace{[f(x)]^2 - [g(x)]^2}^{[OR]^2 - [IR]^2}\} dx}$$

Example 3.

1. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$, $x = 2y$ about the x-axis.



Cross-sectional through x is a washer



$IR = \frac{x}{2}$

$OR = \sqrt{x}$

Points of intersection:

$$\left(\frac{x}{2}\right)^2 = \left(\sqrt{x}\right)^2$$

$$\frac{x^2}{4} = x \longrightarrow \frac{x^2}{4} - x = 0 \longrightarrow x\left(\frac{x}{4} - 1\right) = 0$$

$$\boxed{x_1 = 0,} \quad \boxed{x_2 = 4}$$

$$0 \le x \le 4$$

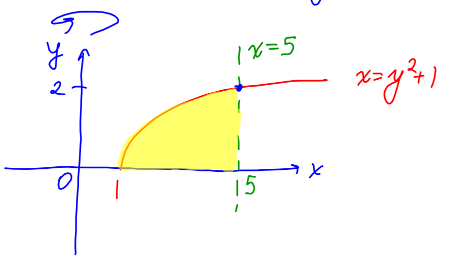
$$V_x = \pi \int_0^4 \left[\left(\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2 \right] dx$$

$$= \pi \int_0^4 \left(x - \frac{x^2}{4} \right) dx = \pi \left(\frac{x^2}{2} - \frac{x^3}{12} \right)\Bigg|_0^4 = \boxed{\pi\left(\frac{\overset{16}{\cancel{4^2}}}{2} - \frac{\overset{64}{\cancel{4^3}}}{12} \right)}$$

2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the y-axis.

x-axis

$$y = \sqrt{x-1} \longrightarrow y^2 = x - 1 \longrightarrow x = y^2 + 1$$



Point of intersection

$$x = y^2 + 1 \quad , \quad x = 5$$

$$5 = y^2 + 1 \longrightarrow y^2 = 4$$

$$\boxed{y = 2} \ (y \geq 0 \text{ only})$$

$$0 \leq y \leq 2$$

$$OR = 5$$

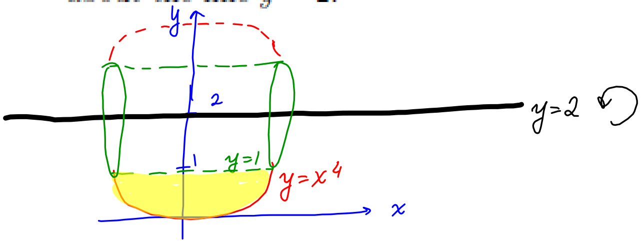
$$IR = y^2 + 1$$

$$V_y = \pi \int_0^2 \left(25 - \underbrace{(y^2+1)^2}_{y^4 + 2y^2 + 1}\right) dy = \pi \int_0^2 \left(25 - y^4 - 2y^2 - 1\right) dy$$

$$= \pi \left(24y - \frac{y^5}{5} - 2\frac{y^3}{3}\right)_0^2$$

$$= \pi \left(24(2) - \frac{2^5}{5} - 2\frac{2^3}{3}\right) = \boxed{\pi\left(48 - \frac{32}{5} - \frac{16}{3}\right)}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = x^4$, $y = 1$ about the line $y = 2$.



$y=2$ is parallel to the x-axis

$\underline{\text{integrate for } x.}$

Points of intersection:

$x^4 = 1$

$x = \pm 1$

$\boxed{-1 \le x \le 1}$

$IR = 1$

$OR = 2 - x^4$

$$V_{(y=2)} = \pi \int_{-1}^{1} \left((2-x^4)^2 - 1 \right) dx$$

$$= \pi \int_{-1}^{1} \left(4 - 4x^4 + x^8 - 1 \right) dx = \pi \int_{-1}^{1} \left(3 - 4x^4 + x^8 \right) dx$$

$$= \pi \left(3x - \frac{4x^5}{5} + \frac{x^9}{9} \right)\Big|_{-1}^{1} = \boxed{\pi \left(3 - \frac{4}{5} + \frac{1}{9} - \left(3(-1) - \frac{4}{5}(-1) + \frac{1}{9}(-1) \right) \right)}$$