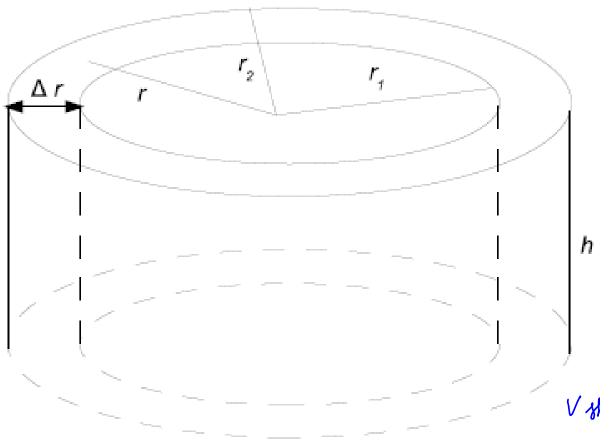
Section 7.3 **Volumes by cylindrical shells**

Lets find the volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (see Fig.1).

$V_{cylinder} = \pi r^2 h$



$V_{shell} = \pi h (r_2^2 - r_1^2)$

$= \pi h (r_2 - r_1)(r_2 + r_1)$

$a^2 - b^2 = (a-b)(a+b)$

$r_2 - r_1 = \Delta r$

$= 2\pi h \frac{r_2 + r_1}{2} \Delta r$

$\frac{r_2 + r_1}{2} = r$ average radius.

$V_{shell} = 2\pi h \, r \, \Delta r$

Fig.1

V can be calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

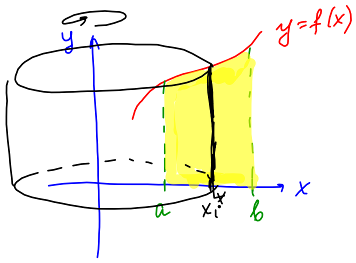
$$V = V_2 - V_1 = \pi h (r_2^2 - r_1^2) = 2\pi h \frac{r_2 + r_1}{2}(r_2 - r_1)$$

Let $\Delta r = r_2 - r_1$, $r = (r_2 + r_1)/2$, then the volume of a cylindrical shell is

$$\boxed{V = 2\pi r h \Delta r}$$

$V =$[circumference][height][thickness]
$V = 2\pi$[average radius][height][thickness]

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.



cylinder generated by rotating the line $x = x_i^*$ about the y-axis.

radius $= x_i^*$

height $= f(x_i^*)$

Volume $= V_i = 2\pi$ [radius][height] Δx

$= 2\pi x_i^* f(x_i^*) \Delta x$

$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} V_i$

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < \dots < x_n = b$ and let x_i^* be the midpoint of $[x_{i-1}, x_i]$, that is $x_i^* = (x_{i-1} + x_i)/2$. If the rectangle with base $[x_{i-1}, x_i]$

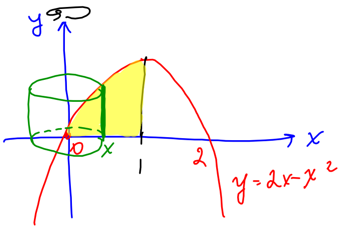
and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average raduis x_i^*, height $f(x_i^*)$, and thikness $\Delta x_i = x_i - x_{i-1}$, so its volume is $V_i = 2\pi x_i^* f(x_i^*) \Delta x_i$.

The approximation to the volume V of S is $V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

Thus, the volume of S is

$$V_Y = \lim_{\|P\| \to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x_i = \boxed{2\pi \int_a^b \overbrace{x f(x)}^{\text{[radius][height]}} dx = V_y}$$

Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 0$ (x-axis), ~~$x = 0$, $x = 1$~~ $0 \le x \le 1$ about the y-axis.



$$V_y = 2\pi \int_0^1 [\text{radius}][\text{height}]\,dx$$

Take $0 \le x \le 1$, vertical line through x, rotate the line about the y-axis.

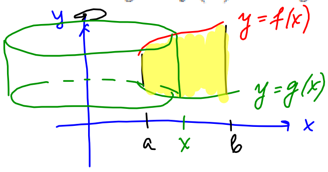
$[\text{radius}] = x$

$[\text{height}] = 2x - x^2$

$$V_y = 2\pi \int_0^1 x(2x - x^2)\,dx = 2\pi \int_0^1 (2x^2 - x^3)\,dx$$

$$= 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4} \right)\Big|_0^1 = \boxed{2\pi\left(\frac{2}{3} - \frac{1}{4} \right)}$$

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b $[f(x) \geq g(x)$ and $0 \leq a < b]$ is

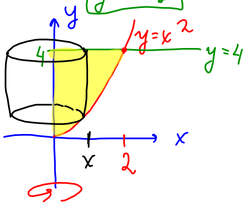


radius = x

height = f(x) - g(x)

$$\boxed{V_Y = 2\pi \int_a^b x[f(x) - g(x)]dx}$$

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, $\underbrace{\boxed{x = 0}}_{y\text{-axis}}$ about the y-axis, $x > 0$.



$0 \leq x \leq 2$

$[\text{radius}] = x$

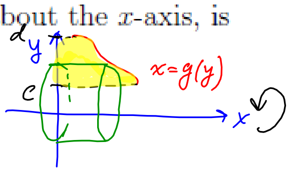
$[\text{height}] = 4 - x^2$

$$V_y = 2\pi \int_0^2 x(4 - x^2)\,dx = 2\pi \int_0^2 (4x - x^3)\,dx$$

$$= 2\pi \left(\frac{4x^2}{2} - \frac{x^4}{4} \right)_0^2$$

$$= \boxed{2\pi \left(\frac{16}{2} - \frac{16}{4} \right)} = \boxed{8\pi}$$

The method of cylindrical shells also allows us to compute volumes of revolution about the x-axis. If we interchange the roles of x and y in the formula for the volume, then the volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is



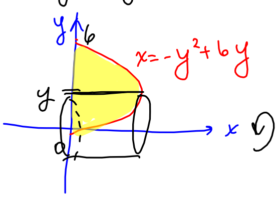
radius = y
height = g(y)

$$V_X = 2\pi \int_c^d \underbrace{y}_{[\text{radius}]}\underbrace{g(y)}_{[\text{height}]}dy$$

Example 3. Find the volume of the solid obtained by rotating the region bounded by $y^2 - 6y + x = 0$, $\boxed{x = 0}$ about the x-axis.

y-axis

$x = -y^2 + 6y$



Integrate for y.

$-y^2 + 6y = 0 \rightarrow y(6-y) = 0$

$y_1 = 0, \quad y_2 = 6$

$0 \le y \le 6$

radius = y
height = $-y^2 + 6y$

$$V = 2\pi \int_0^6 y(-y^2+6y)\,dy = 2\pi \int_0^6 (-y^3 + 6y^2)\,dy$$

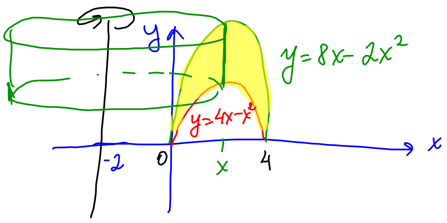
$$= 2\pi\left(-\frac{y^4}{4} + \frac{6y^3}{3}\right)_0^6 = 2\pi\left(-\frac{6^4}{4} + \frac{6(6^3)}{3}\right)_0^6$$

$$= \boxed{2\pi\left(\frac{1296}{3} - \frac{1296}{4}\right)}$$

The volume of the solid generated by rotating the region bounded by $x = g_1(y)$, $x = g_2(y)$, $y = c$, and $y = d$, about the x-axis, assuming that $g_2(y) \geq g_1(y)$ for all $c \leq x \leq d$, is

$$\boxed{V_X = 2\pi \int_c^d y[g_2(y) - g_1(y)]dy}$$

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$, $y = 8x - 2x^2$ about $x = -2$.



Integrate for x.

$0 \leq x \leq 4$

$[\text{radius}] = 2 + x$

$[\text{height}] = 8x - 2x^2 - (4x - x^2)$

$= 4x - x^2$

$$V = 2\pi \int_0^4 [\text{radius}][\text{height}]\,dx = 2\pi \int_0^4 (2+x)(4x - x^2)\,dx$$

$$= 2\pi \int_0^4 (8x - 2x^2 + 4x^2 - x^3)\,dx = 2\pi \int_0^4 (8x + 2x^2 - x^3)\,dx$$

$$= 2\pi \left(\frac{8x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} \right)\Bigg|_0^4$$

$$= \boxed{2\pi \left(\frac{8(16)}{2} + \frac{2(64)}{2} - \frac{256}{4} \right)}$$