Chapter 7. **Applications of integration**
Section 7.4 **Work**

Mechanical work is the amount of energy transferred by a force.

If an object moves along a straight line with position function $s(t)$, then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion

$$F = ma = m\frac{d^2 s}{dt^2}$$

In case of constant acceleration, the force F is also constant and the work done is defined to be the product of the force F and the distance d that the object moves

$$\boxed{W = Fd, \text{ work=force } \times \text{ distance}}$$

Mechanical units in the U.S. customary and SI metric systems

Unit	**U.S. customary system**	**SI metric system**
distance	ft	m
mass	$slug$	kg
force	lb	$N = kg \cdot m/sec^2$
work	$ft - lb$	$J = N \cdot m$
g(Earth)	$32ft/sec^2$	$9.81m/sec^2$

Example 1.

1. Find the work done in pushing a car a distance of 8 m while exerting a constant force of 900 N.

$$\boxed{W = Fd = (8)(900) = 7200 \ (J)}$$

2. How much work is done by a weightlifter in raising a 60-kg barbell from the floor to the height of 2 m?

$F = mg = 60(9.8)$

$W = Fd = \boxed{60(9.8)(2) \ (J)}$

What happens if the force is variable?

Problem The object moves along the x-axis in the positive direction from $x = a$ to $x = b$ and at each point x between a and b a force $f(x)$ acts on the object, where f is continuous function. Find the work done in moving the object from a to b.

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < \ldots < x_n = b$ and let $\Delta x_i = x_i - x_{i-1}$, and let x_i^* is in $[x_{i-1}, x_i]$. Then the force at x_i^* is $f(x_i^*)$. If $\|P\|$ is small, then Δx_i is small, and since f is continuous, the values of f do not change very much on $[x_{i-1}, x_i]$. In other words f is almost a constant on the interval and so work W_i that is done in moving the particle from x_{i-1} to x_i is $W_i \approx f(x_i^*)\Delta x_i$. We can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This approximation becomes better and better as $\|P\| \to 0$.

Therefore, we define the work done in moving the object from a to b as

$$W = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \boxed{\int_a^b f(x)dx = W}$$

Example 2. When a particle is at a distance x meters from the origin, a force of $\cos(\pi x/3)$ N acts on it. How much work is done by moving the particle from $x = 1$ to $x = 2$.

$f(x) = \cos\frac{\pi x}{3}$, $1 \le x \le 2$

$$W = \frac{3}{\pi}\int_1^2 \frac{\pi}{3}\cos\frac{\pi x}{3}\,dx = \left| \begin{array}{l} u = \frac{\pi x}{3} \\ du = \frac{\pi}{3}dx \\ 1 \to \frac{\pi}{3} \\ 2 \to \frac{\pi(2)}{3} = \frac{2\pi}{3} \end{array} \right| = \frac{3}{\pi}\int_{\pi/3}^{2\pi/3} \cos u\,du$$

$$= \frac{3}{\pi}\sin u\Big]_{\pi/3}^{2\pi/3} = \frac{3}{\pi}\left(\sin\frac{2\pi}{3} - \sin\frac{\pi}{3}\right)$$

$$= \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \boxed{0}$$

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is proportional to x

$$f(x) = kx$$

where k is a positive constant (the **spring constant**).

Example 3. Suppose that 2 J of work are needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch it from 35 cm to 40 cm?

convert cm into m
inch into ft

natural length $= 30 \longrightarrow 0$

$42(cm) \longrightarrow 42-30 = 12(cm) = 0.12(m)$

$f(x) = kx$, k is an unknown constant.

$$2 = \int_0^{0.12} kx\,dx$$

$$2 = k\frac{x^2}{2}\Big]_0^{0.12}$$

$$2 = \frac{k}{2}(0.0144) \longrightarrow k = \frac{4}{0.0144} = \frac{2500}{9}$$

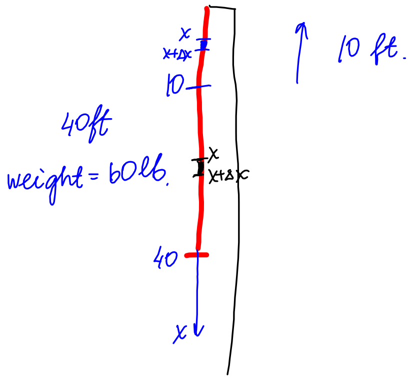
$$\boxed{f(x) = \frac{2500}{9}x}$$

$35\,cm \longrightarrow 35-30 = 5(cm) = 0.05(m)$

$40\,cm \longrightarrow 40-30 = 10(cm) = 0.1(m)$

$$W = \int_{0.05}^{0.1} \frac{2500}{9}x\,dx = \ldots$$

Example 4. A uniform cable hanging over the edge of a tall building is 40 ft long and weights 60 lb. How much work is required to pull 10 ft of the cable to the top?



Work $= \int_a^b (\text{force})(\text{distance})\, dx$

1. $0 \le x \le 10$

Part of the cable between x and $x+\Delta x$

[force] = [weight of the part] $= \frac{60}{40}\Delta x = \frac{3}{2}\Delta x$

[distance traveled] $= x$

2. $10 \le x \le 40$.

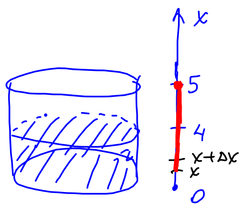
Part of the cable between x and $x+\Delta x$

[weight] $= \frac{3}{2}\Delta x$

[distance traveled] $= 10$

$$\boxed{W = \int_0^{10} \frac{3}{2} x\, dx + \int_{10}^{40} \frac{3}{2}(10)\, dx}$$

Example 5. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all the water out over the side?



$d = 24$ ft, $0 \le x \le 4$

1. Take a "slice" of water between x and $x+\Delta x$
2. Find the weight of the "slice"
3. Find the distance travelled by the "slice"

[distance traveled] $= 5 - x$

62.5 lb/ft^3

[weight] = [volume of the slice]*[weight of the water]

slice = cylinder with base radius 12 and height Δx

volume $= \pi(12^2)\Delta x = 144\pi\Delta x$

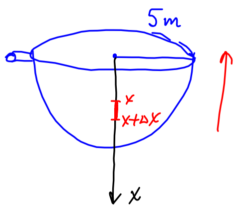
[weight] $= 144(62.5)\pi\Delta x$

$$W = \int_0^4 144(62.5)\pi(5-x)\, dx = 144(62.5)\pi\left(5x - \frac{x^2}{2}\right)\Big|_0^4$$

$$= \boxed{144(62.5)\pi\left(20 - \frac{16}{2}\right)}$$

The tank in a shape of hemisphere with radius 5m is full of water. Find the work done in pumping all the water to the top of the tank. Use the fact that water ~~weighs~~ [redacted].

density is $10^3\ kg/m^3$

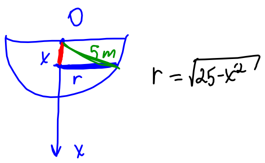


$0 \le x \le 5$.

Take a slice of water between x and $x+\Delta x$

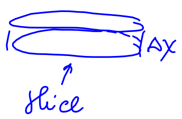
distance traveled $= x$

weight of the slice $= ? = (\text{volume}) * (10^3) * (9.8)$



$$\text{Volume} = \pi r^2 \Delta x$$
$$= \pi(25 - x^2)\Delta x$$

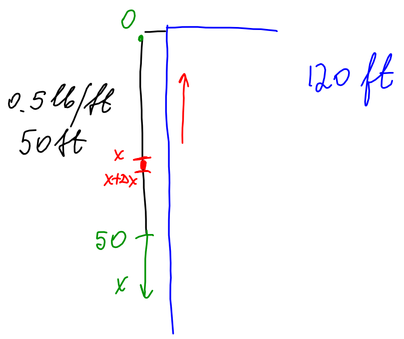
$$\text{weight} = \pi(25-x^2)\Delta x\ (10^3)(9.8)$$



$$W = \int_0^5 x\,\pi(25-x^2)(10^3)(9.8)\,dx$$

$$= \pi(10^3)(9.8)\int_0^5 x(25-x^2)\,dx = \ldots$$

5. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the rope to the top of the building?



Take a small piece of the rope between x and $x+\Delta x$, where $0 \le x \le 50$.

weight $= 0.5\,\Delta x$

distance traveled $\approx x$

$$W = \int_0^{50} 0.5\,x\,dx = 0.5\,\frac{x^2}{2}\Big]_0^{50}$$

$$= \boxed{0.5\,\frac{2500}{2}}$$