

Section 7.5 Average value of a function

Let us try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b-a)/n$ and choose points x_i^* in successive subintervals. Then the average of the numbers $f(x_1^*), f(x_2^*), \dots, f(x_n^*)$, is

$$\frac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{n}$$

Since $n = (b-a)/\Delta x$,

$$\frac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a} (f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x)$$

The limiting value as $n \rightarrow \infty$ is

$$\lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=1}^n f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x) dx$$

We define the **average value of f on the interval $[a, b]$** as

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

Example 1. Find the average value of $f(x) = \sin^2 x \cos x$ on $[\pi/4, \pi/2]$.

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{\frac{\pi}{2} - \frac{\pi}{4}} \int_{\pi/4}^{\pi/2} \sin^2 x \cos x \, dx \\ &= \frac{4}{\pi} \int_{\pi/4}^{\pi/2} \sin^2 x \cos x \, dx \quad \left\{ \begin{array}{l} u = \sin x \\ du = \cos x \, dx \\ \frac{\pi}{4} \rightarrow \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \\ \frac{\pi}{2} \rightarrow \sin \frac{\pi}{2} = 1 \end{array} \right. \\ &= \frac{4}{\pi} \int_{\frac{\sqrt{2}}{2}}^1 u^2 \, du = \frac{4}{\pi} \left[\frac{u^3}{3} \right]_{\frac{\sqrt{2}}{2}}^1 \\ &= \boxed{\frac{4}{3\pi} \left(1 - \left(\frac{\sqrt{2}}{2} \right)^3 \right)} \end{aligned}$$

Example 2. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

$$f_{\text{ave}} = \frac{1}{b-0} \int_0^b (2+6x-3x^2) dx$$

$$= \frac{1}{b} \left(2x + \frac{3bx^2}{2} - \frac{3x^3}{3} \right)_0^b$$

$$= \frac{1}{b} (2b + 3b^2 - b^3) = \boxed{2 + 3b - b^2 = 3}$$

$$-1 + 3b - b^2 = 0$$

$$b^2 - 3b + 1 = 0$$

$$b_1 = \frac{3 + \sqrt{3^2 - 4}}{2}$$

$$\boxed{b_1 = \frac{3 + \sqrt{5}}{2}}$$

$$\boxed{b_2 = \frac{3 - \sqrt{5}}{2}}$$

Mean value theorem for integrals If f continuous on $[a, b]$, then there exist a number c in $[a, b]$ such that

$$\int_a^b f(x) dx = f(c)(b-a)$$

area below the graph of $f(x)$
area of the rectangle

The geometric interpretation of this theorem for *positive* functions $f(x)$, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as a region under the graph of f from a to b .

Example 3. Find the average value of the function $f(x) = 4 - x^2$ on the interval $[0, 2]$. Find c ($0 \leq c \leq 2$) such that $f_{ave} = f(c)$.

$$f_{ave} = \frac{1}{2-0} \int_0^2 (4-x^2) dx$$

$$= \frac{1}{2} \left(4x - \frac{x^3}{3} \right)_0^2$$

$$= \frac{1}{2} \left(8 - \frac{8}{3} \right) = \frac{1}{2} \frac{16}{3} = \boxed{\frac{8}{3} = f_{ave}}$$

$$f(c) = 4 - c^2$$

$$4 - c^2 = \frac{8}{3} \quad (f(c) = f_{ave})$$

$$c^2 = 4 - \frac{8}{3}$$

$$c^2 = \frac{4}{3}$$

$$\boxed{c_1 = \frac{2}{\sqrt{3}}}$$

$$\cancel{c_2 = -\frac{2}{\sqrt{3}}} \quad 0 \leq c \leq 2$$