Section 8.1 **Integration by parts**

$\int u v' dx = uv - \int u' v dx$

$u = u(x)$

$v = v(x)$

The formula for integration by parts for indefinite integrals is

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

The formula for integration by parts for definite integrals is

$$\int_a^b f(x)g'(x)dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

Example 1. Find the integral.

1. $\int x \cos 3x \, dx$

$\int x^n \cos ax \, dx$ $\quad u = x^n \quad v' = \begin{cases} \cos ax \\ \sin bx \\ e^{cx} \end{cases}$

$\int x^n \sin bx \, dx$

$\int x^n e^{cx} \, dx$ $\quad u' = nx^{n-1} \quad v = \begin{cases} \frac{1}{a} \sin ax \\ -\frac{1}{b} \cos bx \\ \frac{1}{c} e^{cx} \end{cases}$

$u = x \quad v' = \cos 3x$

$u' = 1 \quad v = \int \cos 3x \, dx$

$v = \frac{1}{3} \sin 3x$

$uv - \int u'v dx$

$= x \frac{1}{3} \sin 3x - \int 1 \frac{1}{3} \sin 3x \, dx$

$= \frac{1}{3} x \sin 3x - \frac{1}{3} \int \sin 3x \, dx = \frac{1}{3} x \sin 3x - \frac{1}{3} \frac{-1}{3} \cos 3x + C$

$= \boxed{\frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C}$

2. $\int \ln x \, dx$

$\int x^n \ln(ax) dx = \quad u = \ln(ax) \quad v' = x^n$

$u' = \frac{1}{x} \quad v = \frac{x^{n+1}}{n+1}$

$\int 1 \cdot \ln x \, dx \quad u = \ln x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$uv - \int u'v \, dx$

$= x \ln x - \int \frac{1}{x} x \, dx = x \ln x - \int dx = \boxed{x \ln x - x + C}$

3. $\int_0^1 (t^2 + 2t + 3)e^t dt$

$u = t^2 + 2t + 3 \quad v' = e^t$

$u' = 2t + 2 \quad v = e^t$

$= (t^2 + 2t + 3)e^t \Big]_0^1 - \int_0^1 (2t+2) e^t \, dt$

$u = 2t + 2 \quad v' = e^t$

$u' = 2 \quad v = e^t$

$= (1+2+3)e - 3e^0 - \left\{ (2t+2)e^t \Big]_0^1 - \int_0^1 2 e^t \, dt \right\}$

$= 6e - 3 - \left\{ (2+2)e - 2e^0 - 2e^t \Big]_0^1 \right\} = 6e - 3 - \left(4e - 2 - 2(e - e^0)\right)$

$= 6e - 3 - 4e + \cancel{2} + 2e - \cancel{2} = \boxed{4e - 3}$

4. $\int \sin^{-1} x \, dx$

$\left| \begin{array}{ll} u = \sin^{-1} x & v' = 1 \\ u' = \frac{1}{\sqrt{1-x^2}} & v = x \end{array} \right|$ $uv - \int u'v\,dx$ $= x \sin^{-1} x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \, dx$ $\left| \begin{array}{l} q = 1 - x^2 \\ dq = -2x\,dx \end{array} \right.$

$= x \sin^{-1} x + \frac{1}{2} \int \frac{dq}{\sqrt{q}} = x \sin^{-1} x + \frac{1}{2} \int q^{-1/2} \, dq$

$= x \sin^{-1} x + \cancel{\frac{1}{2}} \frac{q^{1/2}}{\cancel{1/2}} + C$

$= \boxed{x \sin^{-1} x + (1-x^2)^{1/2} + C}$

5. $\int e^x \cos x \, dx$ $= \left| \begin{array}{ll} u = \cos x & v' = e^x \\ u' = -\sin x & v = e^x \end{array} \right|$ $uv - \int u'v\,dx$ $= \cos x \, e^x - \int (-\sin x) e^x \, dx$

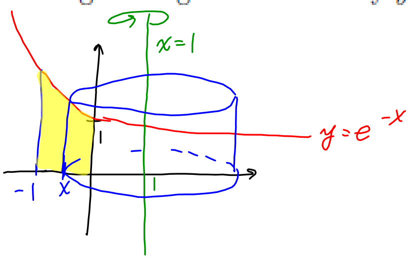
$= \cos x \, e^x + \int \sin x \, e^x \, dx$ $\left| \begin{array}{ll} u = \sin x & v' = e^x \\ u' = \cos x & v = e^x \end{array} \right|$

$\int e^x \cos x \, dx = \cos x \, e^x + \sin x \, e^x - \int \cos x \, e^x \, dx$

$\frac{2 \int e^x \cos x \, dx}{2} = \frac{\cos x \, e^x + \sin x \, e^x}{2}$

$\int e^x \cos x \, dx = \boxed{\frac{1}{2} (\cos x \, e^x + \sin x \, e^x) + C}$

Example 2. Use the method~~s~~ of cylindrical shells to find the volume of a solid generated by rotating the region bounded by $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$ about $x = 1$.



$-1 \le x \le 0$

$[\text{height}] = e^{-x}$

$[\text{radius}] = 1 - x$

$$V = 2\pi \int_{-1}^{0} (1-x)e^{-x}\,dx \quad \left|\; \begin{array}{ll} u = 1-x & v' = e^{-x} \\ u' = -1 & v = -e^{-x} \end{array} \right|$$

$$= 2\pi\left((1-x)(-e^{-x}) \Big]_{-1}^{0} - \int_{-1}^{0} (+1)(+e^{-x})\,dx \right)$$

$$= 2\pi\left(1(-e^{0}) - 2(-e^{-(-1)}) + (+e^{-x}) \Big]_{-1}^{0} \right)$$

$$= 2\pi\left(-1 + 2e + e^{0} - e^{-(-1)} \right)$$

$$= \boxed{2\pi e}$$