Section 8.2 **Trigonometric integrals**

## How to evaluate $\int \sin^m x \cos^n x\, dx$

(a) if $n = 2k+1$ ($n$ is odd) Power of cosine is odd, save one cosine factor and use $\boxed{\cos^2 x = 1 - \sin^2 x}$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x (\cos^2 x)^k \cos x\, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x\, dx$$

Then substitute $\boxed{u = \sin x}$

Power of sine is odd

(b) if $m = 2s+1$ ($m$ is odd), save one sine factor and use $\boxed{\sin^2 x = 1 - \cos^2 x}$ to express the remaining factors in terms of cosine:

$$\int \sin^{2s+1} x \cos^n x\, dx = \int (\sin^2 x)^s \cos^n x \sin x\, dx = \int (1 - \cos^2 x)^s \cos^n x \sin x\, dx$$

Then substitute $\boxed{u = \cos x}$

**Example 1.** Evaluate the integral.

1. $\int \cos^3 x\, dx = \int \cos x \underbrace{\cos^2 x}_{1-\sin^2 x} dx = \int \cos x (1 - \sin^2 x)\, dx$ $\left| \begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array} \right|$

$= \int (1-u^2)\, du = u - \dfrac{u^3}{3} + C$

$= \boxed{\sin x - \dfrac{\sin^3 x}{3} + C}$

2. $\int \sin^5 x \cos^4 x\, dx = \int \sin x \sin^4 x \cos^4 x\, dx = \int \sin x \big(\underbrace{\sin^2 x}_{1-\cos^2 x}\big)^2 \cos^4 x\, dx$

$= \int \sin x (1-\cos^2 x)^2 \cos^4 x\, dx$ $\left| \begin{array}{l} u = \cos x \\ du = -\sin x\, dx \end{array} \right|$

$= -\int (1-u^2)^2 u^4\, du = -\int (1 - 2u^2 + u^4) u^4\, du$

$= -\int (u^4 - 2u^6 + u^8)\, du = -\dfrac{u^5}{5} + 2\dfrac{u^7}{7} - \dfrac{u^9}{9} + C$

$= \boxed{-\dfrac{\cos^5 x}{5} + 2\dfrac{\cos^7 x}{7} - \dfrac{\cos^9 x}{9} + C}$

3. $\int \sin^3 \dfrac{x}{2} \cos^5 \dfrac{x}{2}\, dx = \int \sin\dfrac{x}{2} \underbrace{\sin^2 \dfrac{x}{2}}_{1-\cos^2\frac{x}{2}} \cos^5 \dfrac{x}{2}\, dx = -2\int \cancel{\tfrac{1}{2}}\sin\dfrac{x}{2}\left(1 - \cos^2\dfrac{x}{2}\right)\cos^5\dfrac{x}{2}\, dx$ $\left| \begin{array}{l} u = \cos\frac{x}{2} \\ du = -\frac{1}{2}\sin\frac{x}{2}\, dx \end{array} \right.$

$= -2\int (1-u^2) u^5\, du = -2\int (u^5 - u^7)\, du$

$= -1\cancel{2}\left(-\dfrac{u^6}{3\cancel{6}} + \dfrac{u^8}{4\cancel{8}}\right) + C$

$= \boxed{-\dfrac{\cos^6 \frac{x}{2}}{3} + \dfrac{\cos^8 \frac{x}{2}}{4} + C}$

(c) if both $m$ and $n$ are even, use the half-angle identities

$$\boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x)} \quad \boxed{\cos^2 x = \frac{1}{2}(1 + \cos 2x)}$$

It is sometimes useful to use the identity

$$\boxed{\sin x \cos x = \frac{1}{2}\sin 2x}$$

$\boxed{\int \cos ax\, dx = \frac{1}{a}\sin ax + C}$

**Example 2.** Evaluate each of the following integrals

1. $\int_0^{\pi/2} \sin^2 3x\, dx = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 6x)\, dx = \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right)_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{6}\underbrace{\sin \frac{\cancel{6\pi}^{3\pi}}{\cancel{2}}}_{=0} - 0 + \frac{1}{6}\cancel{\sin 0}^{0}\right)$

$= \boxed{\frac{\pi}{4}}$

2. $\int \cos^4 x\, dx = \int (\overbrace{\cos^2 x}^{\frac{1}{2}(1+\cos 2x)})^2 dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$

$= \frac{1}{4}\int (1+\cos 2x)^2 dx = \frac{1}{4}\int \left(1 + 2\cos 2x + \overbrace{\cos^2 2x}^{\frac{1}{2}(1+\cos 4x)}\right) dx$

$= \frac{1}{4}\int \left(1 + 2\cos 2x + \overbrace{\frac{1}{2}(1+\cos 4x)}^{1/2 + 1/2 \cos 4x}\right) dx$

$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$

$= \boxed{\frac{1}{4}\left(\frac{3}{2}x + \cancel{2}\,\frac{1}{\cancel{2}}\sin 2x + \cancel{\frac{1}{2}\cdot\frac{1}{4}}^{1/8}\sin 4x\right) + C}$

3. $\int_0^{\pi/2} \sin^2 x \cos^2 x\, dx$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$= \int_0^{\pi/2} \left(\frac{1}{2}\sin 2x\right)^2 dx = \frac{1}{4}\int_0^{\pi/2} \overbrace{\sin^2 2x}^{\frac{1}{2}(1-\cos 4x)}\, dx$

$= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x)\, dx = \frac{1}{8}\int_0^{\pi/2}(1 - \cos 4x)\, dx$

$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right)_0^{\pi/2}$

$= \frac{1}{8}\left(\frac{\pi}{2} - \frac{1}{4}\cancel{\sin \frac{4\pi}{2}}^{0} + \frac{1}{4}\cancel{\sin 0}^{0}\right) = \boxed{\frac{\pi}{16}}$

**How to evaluate** $\int \tan^m x \sec^n x \, dx$

(a) if the power of secant is even $n = 2k$, save a factor of $\sec^2 x$ and use $\boxed{\sec^2 x = 1 + \tan^2 x}$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $\boxed{u = \tan x}$

**Example 3.**

1. $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$
$0 \to \tan 0 = 0$
$\frac{\pi}{4} \to \tan \frac{\pi}{4} = 1$

$= \int_0^1 u^4 \, du = \frac{u^5}{5} \Big]_0^1 = \boxed{\frac{1}{5}}$

$\tan^2 x = \sec^2 x - 1$

2. $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \int \sec^2 x \, dx - \int 1 \, dx = \boxed{\tan x - x + C}$

3. $\int \tan^4 x \, dx = \int (\tan^2 x)^2 dx = \int (\sec^2 x - 1)^2 \, dx = \int (\sec^4 x - 2\sec^2 x + 1) \, dx$

$= \int \sec^4 x \, dx - 2 \int \sec^2 x \, dx + \int 1 \, dx$

$= \frac{\tan^3 x}{3} + \tan x - 2\tan x + x + C$

$= \boxed{\frac{\tan^3 x}{3} - \tan x + x + C}$

4. $\int \sec^4 x\, dx = \int \sec^2 x \overbrace{\sec^2 x}^{\tan^2 x+1} dx = \int \sec^2 x(\tan^2 x+1)dx = \int \sec^2 x \tan^2 x\,dx + \int \sec^2 x\, dx$

$\tan x$

$u = \tan x$
$du = \sec^2 x\, dx$

$= \int u^2 du + \tan x = \frac{u^3}{3} + \tan x + C$

$= \boxed{\frac{\tan^3 x}{3} + \tan x + C}$

5. $\int \tan^3 x\, dx = \int \tan x \overbrace{\tan^2 x}^{\sec^2 x - 1} dx = \int \tan x(\sec^2 x - 1)\, dx$

$= \int \tan x \sec^2 x\, dx - \int \tan x\, dx$

$u = \tan x$
$du = \sec^2 x\, dx$

$= \int u\, du = \frac{u^2}{2} + C$

$= \frac{\tan^2 x}{2} + C$

$= -\int \frac{\sin x}{\cos x}\, dx$

$v = \cos x$
$dv = -\sin x\, dx$

$= \int \frac{dv}{v} = \ln|v| + C$

$= \ln|\cos x| + C$

$\int \tan^3 x\, dx = \boxed{\frac{\tan^2 x}{2} + \ln|\cos x| + C}$

$\boxed{\tan^2 x = \sec^2 x - 1}$

(b) if the $\boxed{\text{power of tangent is odd}}$ ($m = 2s+1$), save a factor of $\tan x \sec x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2s+1} x \sec^n x \, dx = \int (\tan^2 x)^s \sec^m x \tan x \sec x \, dx = \int (\sec^2 x - 1)^s \sec^m x \tan x \sec x \, dx$$

Then substitute $\boxed{u = \sec x}$

**Example 4.**

1. $\int \tan^3 x \ \sec^3 x \, dx \;=\; \int (\sec x \tan x)\, \overbrace{\tan^2 x}^{\sec^2 x - 1} \sec^2 x \, dx = \int (\sec x \tan x)(\sec^2 x - 1)\sec^2 x \, dx$

$$\left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right| = \int (u^2-1)u^2 du = \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

2. $\int \sec x \, dx \;=\; \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \quad \left| \begin{array}{l} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x) dx \end{array} \right|$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

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3. $\int \sec^3 x \, dx \;=\; \int \sec x \ \sec^2 x \, dx$

Integrate by parts.

$$\left| \begin{array}{ll} u = \sec x & v' = \sec^2 x \\ u' = \sec x \tan x & v = \tan x \end{array} \right|$$

$uv - \int u'v dx$

$$= \sec x \tan x - \int \sec x \tan x \tan x \, dx$$

$$= \sec x \tan x - \int \sec x \, \overbrace{\tan^2 x}^{\sec^2 x - 1} \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx$$

$$\frac{2\int \sec^3 x \, dx}{2} = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2}$$

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

Integrals of the form

$$\int \cot^m x \csc^n x \, dx$$

can be found by similar methods because of the identity $\boxed{1+\cot^2 x = \csc^2 x.}$

**Example 5.** Find

1. $\int \cot^4 x \csc^4 x \, dx \;=\; \int (\csc^2 x) \cot^4 x \underbrace{\csc^2 x}_{1+\cot^2 x} dx = \int (\csc^2 x) \cot^4 x \, (1+\cot^2 x) \, dx$ $\quad \begin{array}{|l|} u = \cot x \\ du = -\csc^2 x \, dx \end{array}$

$$= -\int u^4 (1+u^2) \, du = -\int (u^4+u^6) \, du = -\frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \boxed{-\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + C}$$

2. $\int \cot^3 x \csc^2 x \, dx \;=\; \int (\cot x \csc x) \underbrace{\cot^2 x}_{\csc^2 x - 1} \csc x \, dx = \int (\cot x \csc x)(\csc^2 x - 1) \csc x \, dx$ $\quad \begin{array}{|l|} u = \csc x \\ du = -\csc x \cot x \, dx \end{array}$

$$= -\int (u^2-1) \, u \, du = -\int (u^3-u) \, du = -\frac{u^4}{4} + \frac{u^2}{2} + C$$

$$= \boxed{-\frac{\csc^4 x}{4} + \frac{\csc^2 x}{2} + C}$$

To evaluate the integrals

$$\text{(a)} \int \sin mx \ \cos nx \ dx$$

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$$\text{(b)} \int \sin mx \ \sin nx \ dx$$

$$\text{(c)} \int \cos mx \ \cos nx \ dx$$

use the corresponding identity:

$$\text{(a)} \sin\alpha \ \cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

$$\text{(b)} \sin\alpha \ \sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\text{(c)} \cos\alpha \ \cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\boxed{\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}}$$

$$\boxed{\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}}$$

**Example 6.**

1. $\int \sin 5x \ \sin 2x \ dx = \int \frac{1}{2}\left(\cos(5x-2x) - \cos(5x+2x)\right) dx$

$\boxed{\int \cos ax \, dx = \frac{1}{a}\sin ax + C}$

$= \frac{1}{2}\int \left(\cos 3x - \cos 7x\right) dx = \boxed{\frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x\right) + C}$

2. $\int \sin 3x \ \cos x \ dx = \int \frac{1}{2}\left(\sin(3x-x) + \sin(3x+x)\right) dx$

$\boxed{\int \sin 6x \, dx = -\frac{1}{6}\cos 6x + C}$

$= \frac{1}{2}\int\left(\sin 2x + \sin 4x\right) dx = \boxed{\frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C}$

3. $\int \cos 3x \ \cos 4x \ dx \overset{\text{identity (c)}}{=} \int \frac{1}{2}\left(\cos(3x-4x)+\cos(3x+4x)\right) dx$

$= \frac{1}{2}\int\left(\overset{\cos(-x)=\cos x}{\cos(-x)}+\cos 7x\right) dx$

$= \frac{1}{2}\int\left(\cos x+\cos 7x\right) dx = \boxed{\frac{1}{2}\left(\sin x+\frac{1}{7}\sin 7x\right)+C}$