Chapter 8. **Techniques of integration**
Section 8.3 **Trigonometric substitution**

Assume that g is one-to-one function (g^{-1} exists). Then

$$\boxed{\int f(x)dx = \int f(g(t))g'(t)dt}$$

$x = g(t)$
$dx = g'(t)dt$

This kind of substitution is called *inverse substitution*.

Table of trigonometric substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin t, -\pi/2 \le t \le \pi/2$	$1 - \sin^2 t = \cos^2 t$
$\sqrt{a^2 + x^2}$	$x = a\tan t, -\pi/2 < t < \pi/2$	$1 + \tan^2 t = \sec^2 t$
$\sqrt{x^2 - a^2}$	$x = a\sec t, 0 \le t \le \pi/2$ or $\pi \le t \le 3\pi/2$	$\sec^2 t - 1 = \tan^2 t$

Example Find

(a) $\int x\sqrt{4 - x^2}dx$

$2\sin t$ $\quad$ $2\cos t$ $\quad$ $2\cos t\,dt$

$x = 2\sin t$
$dx = 2\cos t\,dt$

$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2 t} = \sqrt{4(\underbrace{1-\sin^2 t}_{\cos^2 t})}$

$= \sqrt{4\cos^2 t} = 2\cos t$

$$\boxed{\begin{array}{c}\sqrt{4-x^2} = 2\cos t \\ \cos t = \dfrac{\sqrt{4-x^2}}{2}\end{array}}$$

$= \int \overbrace{2\sin t}^{x}\ \overbrace{2\cos t}^{\sqrt{4-x^2}}\ \overbrace{2\cos t\,dt}^{dx}$

$= 8\int \sin t\cos^2 t\,dt$ $\quad$ $\begin{vmatrix} u = \cos t \\ du = -\sin t\,dt \end{vmatrix}$

$= -8\int u^2 du = -8\dfrac{u^3}{3} + C$

$= -\dfrac{8}{3}\cos^3 t + C$

$= -\dfrac{8}{3}\left(\dfrac{\sqrt{4-x^2}}{2}\right)^3 + C$

$= \boxed{-\dfrac{1}{3}(4-x^2)^{3/2} + C}$

(b) $\int \frac{x^3}{\sqrt{x^2+4}}dx$ =

$$\begin{aligned} & x = 2\tan t \\ & dx = 2\sec^2 t\, dt \\ & \sqrt{x^2+4} = \sqrt{4\tan^2 t+4} = \sqrt{4(\underbrace{\tan^2 t+1}_{\sec^2 t})} \\ & = \sqrt{4\sec^2 t} = 2\sec t \end{aligned}$$

$$= \int \frac{(2\tan t)^3}{\cancel{2\sec t}} \cancel{2}\sec^{\cancel{2}} t\, dt$$

$$= \int 8\tan^3 t \sec t\, dt$$

$$= 8\int (\tan t \sec t)\tan^2 t\, dt$$

$$= 8\int (\tan t \sec t)(\sec^2 t - 1)\, dt$$

$$\begin{aligned} & u = \sec t \\ & du = \tan t \sec t\, dt \end{aligned}$$

$$= 8\int (u^2-1)\, du = 8\left(\frac{u^3}{3} - u\right) + C$$

$$= 8\left(\frac{\sec^3 t}{3} - \sec t\right) + C$$

$$\boxed{= 8\left(\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right) + C}$$

$$\begin{aligned} & \sqrt{x^2+4} = 2\sec t \\ & \sec t = \frac{\sqrt{x^2+4}}{2} \end{aligned}$$

(c) $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{dx}{x^2\sqrt{16\left(x^2-\frac{9}{16}\right)}} = \frac{1}{4}\int \frac{dx}{x^2\sqrt{x^2-\frac{9}{16}}} =$

$$= \frac{1}{4}\int \frac{\cancel{\frac{3}{4}}\,\cancel{\sec t}\,\cancel{\tan t}\,dt}{\frac{9}{16}\sec^{\cancel{2}} t\,\cancel{\frac{3}{4}}\,\cancel{\tan t}}$$

$$= \frac{1}{4}\,\frac{16}{9}\int \frac{dt}{\sec t} = \frac{4}{9}\int \cos t\,dt$$

$$= \frac{4}{9}\sin t + C$$

$$\boxed{= \frac{\cancel{4}}{9}\,\frac{\sqrt{16x^2-9}}{\cancel{4}x} + C}$$

$4x = 3\sec t$

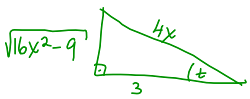
$x = \sqrt{\frac{9}{16}}\sec t = \frac{3}{4}\sec t$

$dx = \frac{3}{4}\sec t\tan t\,dt$

$\sqrt{x^2-\frac{9}{16}} = \sqrt{\frac{9}{16}\sec^2 t - \frac{9}{16}}$

$= \sqrt{\frac{9}{16}\underbrace{(\sec^2 t - 1)}_{\tan^2 t}} = \sqrt{\frac{9}{16}\tan^2 t}$

$= \frac{3}{4}\tan t$



$x = \frac{3}{4}\sec t$

$\sec t = \frac{4x}{3}$

$\cos t = \frac{1}{\sec t} = \frac{3}{4x}$

$$\boxed{\sin t = \frac{\sqrt{16x^2-9}}{4x}}$$

$$(a+b)^2 = a^2+2ab+b^2$$

(d) $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+8}}$

$(x+2)^2 = x^2+4x+4$

1. Complete the square: $\underbrace{x^2+4x}_{a^2+2ab}+8 = (x^2+4x+4)+4 = (x+2)^2+4$

$a = x$ and $2ab = 4x$

$b = \frac{4x}{2a} = \frac{4x}{2x} = 2$

$$= \int \frac{dx}{\sqrt{(x+2)^2+4}}$$

2. ~~Do~~ the appropriate substitution.

$x+2 = 2\tan t$

$x = 2\tan t - 2$

$dx = 2\sec^2 t\, dt$

$\sqrt{(x+2)^2+4} = \sqrt{4\tan^2 t+4} = \sqrt{4\underbrace{(\tan^2 t+1)}_{\sec^2 t}}$

$= \sqrt{4\sec^2 t} = 2\sec t$

$x+2 = 2\tan t \rightarrow \tan t = \frac{x+2}{2}$

$2\sec t = \sqrt{(x+2)^2+4}$

$\sec t = \frac{\sqrt{(x+2)^2+4}}{2}$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}} t\, dt}{\cancel{2}\cancel{\sec t}} = \int \sec t\, dt = \ln|\sec t+\tan t| + C$$

$$\boxed{= \ln\left|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}\right| + C}$$

(e) $\int e^t\sqrt{9-e^{2t}}\,dt$

$u = e^t$
$du = e^t\,dt$

$= \int \sqrt{9-u^2}\,du$

$u = 3\sin z$
$du = 3\cos z\,dz$
$\sqrt{9-u^2} = \sqrt{9-9\sin^2 z} = \sqrt{9(\underbrace{1-\sin^2 z}_{\cos^2 z})}$
$= \sqrt{9\cos^2 z} = 3\cos z$

$= \int 3\cos z \; 3\cos z\,dz$
$= 9\int \cos^2 z\,dz$
$= 9\int \frac{1}{2}(1+\cos 2z)\,dz$
$= \frac{9}{2}\left(z + \frac{1}{2}\underbrace{\sin 2z}_{2\sin z\cos z}\right) + C$

$= \frac{9}{2}(z + \sin z\cos z) + C$

$u = 3\sin z \Rightarrow z = \sin^{-1}\left(\frac{u}{3}\right)$
$\sin z = \frac{u}{3}$
$\sqrt{9-u^2} = 3\cos z$
$\cos z = \frac{\sqrt{9-u^2}}{3}$

$= \frac{9}{2}\left(\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3}\frac{\sqrt{9-u^2}}{3}\right) + C$

$= \boxed{\frac{9}{2}\left(\sin^{-1}\left(\frac{e^t}{3}\right) + \frac{e^t\sqrt{9-e^{2t}}}{9}\right) + C}$