Chapter 8. **Techniques of integration**

Section 8.4 Integration of rational functions by partial fractions

In this section we show how to integrate any rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, $Q(x) = b_0x^m + b_1x^{m-1} + ... + b_m$ by expressing it as a sum of *partial fractions*, that we know how to integrate.

STEP 1. If f is improper ($n \geq m$), then we must divide P into Q by long divisions until a remainder $R(x)$ is obtained. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

STEP 2. Factor the denominator $Q(x)$ as far as possible. It can be shown that any polynomial Q can be factored as a product of *linear factors* of the form $ax + b$ and *irreducible quadratic factors* (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

STEP 3. Express the proper rational function $\dfrac{R(x)}{Q(x)}$ as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

Four cases occur.

CASE I. $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (a_1x + b_1)(a_2x + b_2)...(a_mx + b_m)$$

where no factor is repeated. Then there exist constants A_1, A_2,..., A_m such that

$$f(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_m}{a_mx + b_m}$$

Once the constants A_1, A_2,... , A_m are determined, the evaluation of $\dfrac{R(x)}{Q(x)}$ becomes a routine problem. The next example will illustrate one method for finding these constants.

Example 1. Evaluate $\int_2^4 \frac{4x-1}{x^2+x-2}dx$

- $\frac{4x-1}{x^2+x-2}$ is <u>proper</u>

- Factor $x^2+x-2=(x+2)(x-1)$

- Partial fractions:

$$\frac{4x-1}{x^2+x-2}=\frac{4x-1}{(x+2)(x-1)}=\frac{\cancel{A}^{3}}{x+2}+\frac{\cancel{B}^{1}}{x-1}, \quad A, B \text{ are unknown const.}$$

$$\frac{4x-1}{\cancel{(x+2)(x-1)}}=\frac{A(x-1)+B(x+2)}{\cancel{(x+2)(x-1)}} \quad \text{drop the denominator}$$

$$4x-1=A(x-1)+B(x+2)$$

$$\boxed{\int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|+C}$$

$x=1$: $4(1)-1=A(0)+B(3) \Rightarrow 3=3B \Rightarrow \boxed{B=1}$

$x=-2$: $4(-2)-1=A(-2-1)+B(0) \Rightarrow -9=-3A \Rightarrow \boxed{A=3}$

$$\int_2^4 \frac{4x-1}{(x+2)(x-1)}dx=\int_2^4\left(\frac{3}{x+2}+\frac{1}{x-1}\right)dx$$

$$=\Big[3\ln|x+2|+\ln|x-1|\Big]_2^4$$

$$=3\ln 6+\ln 3-3\ln 4-\cancel{\ln 1}^{0}=\ln\frac{(6^3)(3)}{4^3}=\boxed{\ln\frac{81}{8}}$$

CASE II. $Q(x)$ **is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $a_1x + b_1$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$, we would use

$$\overset{r>1}{(a_1x+b_1)^r} \longrightarrow \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \dots + \frac{A_r}{(a_1x + b_1)^r}$$

Example 2. Evaluate $\displaystyle\int \frac{5x^2 + 6x + 9}{(x + 1)^2(x - 3)^2}dx$

$$\frac{5x^2+6x+9}{(x+1)^2(x-3)^2} = \frac{\cancel{A}^{\,0}}{x+1} + \frac{\cancel{B}^{\,1/2}}{(x+1)^2} + \frac{\cancel{C}^{\,0}}{x-3} + \frac{\cancel{D}^{\,9/2}}{(x-3)^2}$$

$$\frac{5x^2+6x+9}{\cancel{(x+1)^2(x-3)^2}} = \frac{A(x+1)(x-3)^2 + B(x-3)^2 + C(x-3)(x+1)^2 + D(x+1)^2}{\cancel{(x+1)^2(x-3)^2}}$$

$$5x^2+6x+9 = A(x+1)(x-3)^2 + B(x-3)^2 + C(x-3)(x+1)^2 + D(x+1)^2$$

$x=-1$: $5(-1)^2+6(-1)+9 = A(0) + B(-1-3)^2 + C(0) + D(0)$

$8 = 16B \Rightarrow \boxed{B = 1/2}$

$x=3$: $5(3^2)+6(3)+9 = 0+0+0+D(3+1)^2$

$72 = 16D \Rightarrow D = \frac{72}{16} = \boxed{\frac{9}{2} = D}$

$x=0$: $9 = A(-3)^2 + B(-3)^2 + C(-3) + D$

$9 = 9A + 9\cancel{B}^{\,1/2} - 3C + \cancel{D}^{\,9/2} \Rightarrow 9 = 9A + \frac{9}{2} - 3C + \frac{9}{2}$

$\cancel{9} = 9A - 3C + \cancel{9} \Rightarrow 9A - 3C = 0$

$\boxed{C = 3A}$

$x=1$: $5+6+9 = A(2)(-2)^2 + B(-2)^2 + C(-2)(2)^2 + D(2^2)$

$\frac{20}{4} = \frac{8A + 4B - 8C + 4D}{4}$

$5 = 2A + \cancel{B}^{\,1/2} - 2\cancel{C}^{\,3A} + \cancel{D}^{\,9/2}$

$5 = 2A + \frac{1}{2} - 2\underbrace{(3A)}_{C} + \frac{9}{2}$

$5 = 5 - 4A \Rightarrow \boxed{A = 0}\ \boxed{C = 0}$

$$\boxed{\int \frac{dx}{(ax+b)^2} = \frac{-1}{a}\frac{1}{ax+b} + C}$$

$$\int \frac{5x^2+6x+9}{(x+1)^2(x-3)^2}dx = \int\left(\frac{1}{2}\frac{1}{(x+1)^2} + \frac{9}{2}\frac{1}{(x-3)^2}\right)dx$$

$$= \boxed{-\frac{1}{2}\frac{1}{x+1} - \frac{9}{2}\frac{1}{x-3} + C}$$

CASE III $Q(x)$ contains irreducible quadratic factors none of which is repeated.
If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then the corresponding fraction is

$$\boxed{\underset{b^2-4ac<0}{ax^2+bx+c} \longrightarrow \frac{Ax+B}{ax^2+bx+c}}$$

where A and B are constants to be determined.

The term $\dfrac{Ax+B}{ax^2+bx+c}$ can be integrating by completing the square in the denominator.

Example 3. Find $\displaystyle\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}dx$

$$\frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)} = \frac{\overset{3}{A}x+\overset{-3}{B}}{x^2+1} + \frac{\overset{0}{C}x+\overset{2}{D}}{x^2+2}$$

$$\frac{3x^3-x^2+6x-4}{\cancel{(x^2+1)(x^2+2)}} = \frac{(Ax+B)(x^2+2)+(Cx+D)(x^2+1)}{\cancel{(x^2+1)(x^2+2)}}$$

$$3x^3-x^2+6x-4 = (Ax+B)(x^2+2)+(Cx+D)(x^2+1)$$
$$= Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+D$$

$$3x^3-x^2+6x-4 = x^3(A+C)+x^2(B+D)+x(2A+C)+(2B+D)$$

$$\begin{array}{l} x^3: \\ x^2: \\ x: \\ 1: \end{array}\left\{\begin{array}{l} 3=A+C \\ -1=B+D \\ 6=2A+C \\ -4=2B+D \end{array}\right. \Rightarrow \quad \left\{\begin{array}{l} 3=A+C \\ 6=2A+C \end{array}\right. \quad \left\{\begin{array}{l} -1=B+D \\ -4=2B+D \end{array}\right.$$

$$C=3-A \qquad D=-1-B$$
$$6=2A+3-A \qquad -4=2B+(-1-B)$$
$$\boxed{A=3}\ \boxed{C=0} \qquad -4+1=B \Rightarrow \boxed{B=-3}$$
$$\boxed{D=-1-(-3)=2}$$

$$\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}dx = \int\left(\frac{3x-3}{x^2+1}+\frac{2}{x^2+2}\right)dx$$

$$\boxed{\int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C}$$

$$= \int\left(\frac{3x}{x^2+1}-\frac{3}{x^2+1}+\frac{2}{x^2+2}\right)dx$$

$$= \frac{3}{2}\int\frac{2x\,dx}{x^2+1} - 3\int\frac{dx}{x^2+1} + 2\int\frac{dx}{x^2+2}$$

$$= \left| \begin{array}{l} u=x^2+1 \\ du=2x\,dx \end{array}\right|$$

$$= \frac{3}{2}\int\frac{du}{u} - 3\arctan x + 2\cdot\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}$$

$$= \frac{3}{2}\ln|u| - 3\arctan x + \sqrt{2}\arctan\frac{x}{\sqrt{2}} + C$$

$$= \boxed{\frac{3}{2}\ln|x^2+1| - 3\arctan x + \sqrt{2}\arctan\frac{x}{\sqrt{2}} + C}$$

CASE IV $Q(x)$ contains a repeated irreducible factor.

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction $\dfrac{Ax+B}{ax^2+bx+c}$, the sum

$$\boxed{\underset{b^2-4ac<0}{(ax^2+bx+c)^r} \rightarrow \frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of these terms can be integrated by completing the square and making the tangent substitution.

Example 4. Write out the form of the partial fraction decomposition of the function

$$\frac{x-3}{(x^2+x+1)^2(x^2+2x+4)^2}.$$

Do not determine the numerical values for the coefficients.

$$\boxed{\frac{x-3}{(x^2+x+1)^2(x^2+2x+4)^2}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}+\frac{Ex+F}{x^2+2x+4}+\frac{Gx+H}{(x^2+2x+4)^2}}$$

Example 5. Evaluate $\displaystyle\int \frac{x+4}{(x^2+x+1)^2}\,dx$

$$x^2+x+1 = \underbrace{x^2+2\cdot\tfrac{1}{2}x+\left(\tfrac{1}{2}\right)^2}_{(x+\frac{1}{2})^2}\;\underbrace{-\left(\tfrac{1}{2}\right)^2+1}_{3/4}$$

$$= \int \frac{x+4}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\,dx$$

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan t$$

$$x = \frac{\sqrt{3}}{2}\tan t - \frac{1}{2}$$

$$dx = \frac{\sqrt{3}}{2}\sec^2 t\,dt$$

$$\left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\tan^2 t+\frac{3}{4}$$

$$= \frac{3}{4}(\tan^2 t+1) = \frac{3}{4}\sec^2 t$$

$$= \int \frac{\frac{\sqrt{3}}{2}\tan t-\frac{1}{2}+4}{\left(\frac{3}{4}\sec^2 t\right)^{\cancel{2}}}\;\frac{\sqrt{3}}{2}\,\cancel{\sec^2 t}\,dt$$

$$= \int \frac{\left(\frac{\sqrt{3}}{2}\tan t+\frac{7}{2}\right)\frac{\sqrt{3}}{2}}{\frac{9}{16}\sec^2 t}\,dt = \frac{16}{9}\int \frac{\frac{3}{4}\tan t+\frac{7\sqrt{3}}{4}}{\sec^2 t}\,dt$$

$$= \frac{16}{9}\,\frac{3}{4}\int \frac{\tan t}{\sec^2 t}\,dt + \frac{16}{9}\,\frac{7\sqrt{3}}{4}\int \frac{dt}{\sec^2 t}$$

$$= \frac{4}{3}\int \frac{\sin t}{\cos t}\cos^2 t\,dt + \frac{28\sqrt{3}}{9}\int \cos^2 t\,dt$$

$$= \frac{4}{3}\underbrace{\int \sin t\cos t\,dt}_{\int u\,du} + \frac{28\sqrt{3}}{9}\int \frac{1}{2}(1+\cos 2t)\,dt$$

$u = \sin t$
$du = \cos t\,dt$

$$= \frac{4}{3}\,\frac{u^2}{2} + \frac{14\sqrt{3}}{9}\left(t+\frac{1}{2}\sin 2t\right)+C$$

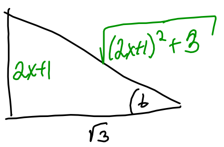
$$= \frac{2}{3}\sin^2 t + \frac{14\sqrt{3}}{9}\Big(t+\underbrace{\tfrac{1}{2}\sin 2t}_{\sin t\cos t}\Big)+C$$

$$\boxed{= \frac{2}{3}\left(\frac{2x+1}{\sqrt{(2x+1)^2+3}}\right)^2 + \frac{14\sqrt{3}}{9}\left(\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{(2x+1)(\sqrt{3})}{(2x+1)^2+3}\right)+C}$$

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan t$$

$$\tan t = \frac{x+1/2}{\sqrt{3}/2} = \frac{2x+1}{\sqrt{3}}$$

$$t = \arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$



$$\sin t = \frac{2x+1}{\sqrt{(2x+1)^2+3}}$$

$$\cos t = \frac{\sqrt{3}}{\sqrt{(2x+1)^2+3}}$$