Chapter 8. **Techniques of integration**
Section 8.9 **Improper integrals**

In this section we extend the conception of a definite integral to the case where the interval is infinite and also to the case where integrand is unbounded.

Definition of an improper integral of type 1 (infinite intervals)

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number)

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

where a is any real number

Example 1. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}dx$ convergent?

Example 2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\displaystyle\int_{2}^{\infty} \frac{dx}{\sqrt{x+3}}$

2. $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9}$

3. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x\sqrt[3]{x-1}}$

4. $\int_{-\infty}^{\infty} (2x^2+x-1)dx$

5. $\int\limits_0^{\infty} \frac{1}{(x+2)(x+3)} dx$

6. $\int\limits_1^{\infty} e^x dx$

7. $\int\limits_{-\infty}^{1} e^x dx$

8. $\int\limits_{-\infty}^{\infty} e^x dx$

9. $\int\limits_{e}^{\infty} \frac{dx}{x(\ln x)^2}$

Definition of an improper integral of type 2 (discontinuous integrands)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int\limits_{a}^{b} f(x)dx = \lim_{t \to b^-} \int\limits_{a}^{t} f(x)dx$$

if this limit exists (as a finite number)

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int\limits_{a}^{b} f(x)dx = \lim_{t \to a^+} \int\limits_{t}^{b} f(x)dx$$

if this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If f has discontinuity at c $(a < c < b)$, and both $\int\limits_{a}^{c} f(x)dx$ and $\int\limits_{c}^{b} f(x)dx$ are convergent, then

$$\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{c} f(x)dx + \int\limits_{c}^{b} f(x)dx$$

Example 3. For what values of p is the integral $\int\limits_{0}^{1} \frac{1}{x^p}dx$ convergent?

Example 4. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\displaystyle\int_{-3}^{0} \frac{dx}{\sqrt{x+3}}$

2. $\displaystyle\int_{0}^{3} \frac{1}{x\sqrt{x}}dx$

3. $\int_{\pi/4}^{\pi/2} \sec^2 x dx$

4. $\int_{0}^{1} \ln x dx$

Comparison theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example 5. Use the Comparison Theorem to determine whether $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent or divergent.