Section 8.9 **Improper integrals**

In this section we extend the conception of a definite integral to the case where the interval is infinite and also to the case where integrand is unbounded.

Definition of an improper integral of type 1 (infinite intervals)

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\boxed{\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx}$$

provided this limit exists (as a finite number)

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\boxed{\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx}$$

provided this limit exists (as a finite number)

The improper integrals in (a) and (b) are called convergent if the limit exist and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we define

$$\boxed{\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx + \lim_{s\to\infty} \int_a^s f(x)dx}$$

where a is any real number

convergent + convergent = convergent
convergent + divergent = divergent
divergent + divergent = divergent

Example 1. For what values of p is the integral $\displaystyle\int_1^\infty \frac{1}{x^p}dx$ convergent?

1. $p>1 \Rightarrow 1-p<0$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\frac{x^{-p+1}}{-p+1}\Big]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{t^{\overbrace{-p+1}^{\text{negative}}}}{-p+1} - \frac{1}{-p+1}\right)$$

$$= \lim_{t\to\infty} \overset{0}{\cancel{\frac{1}{t^{\underbrace{p-1}_{\text{positive}}}(-p+1)}}} - \frac{1}{-p+1} = \frac{1}{p-1} \quad \text{converges}$$

2. $p=1.$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t = \lim_{t\to\infty}\left[\ln|t| - \overset{0}{\cancel{\ln 1}}\right]$$

$$= \lim_{t\to\infty} \ln|t| = \infty \quad \text{diverges}$$

3. $p<1. \Rightarrow 1-p>0$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty}\frac{x^{-p+1}}{-p+1}\Big]_1^t = \lim_{t\to\infty}\left(\overset{\infty}{\cancel{\frac{t^{\overbrace{-p+1}^{\text{positive}}}}{-p+1}}} - \frac{1}{-p+1}\right) = \infty \quad \text{diverges}$$

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \quad \text{converges} \\ \infty, & \text{if } p\le 1 \quad \text{diverges}\end{cases}}$$

Example 2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x+3}}$

$p = \frac{1}{2} < 1$

$\boxed{\text{diverges}}$

$p = 2 > 1$ converges.

2. $\displaystyle\int_{-\infty}^3 \frac{dx}{x^2+9} = \lim_{t\to-\infty} \int_t^3 \frac{dx}{x^2+9} = \lim_{t\to-\infty} \frac{1}{3}\arctan\frac{x}{3}\Big]_t^3$

$= \frac{1}{3}\lim_{t\to-\infty}\left(\underbrace{\arctan\frac{3}{3}}_{\pi/4} - \arctan\frac{t}{3}\right)$

$= \frac{1}{3}\frac{\pi}{4} - \frac{1}{3}\underbrace{\lim_{t\to-\infty}\arctan\frac{t}{3}}_{-\frac{\pi}{2}}$

$= \frac{1}{3}\frac{\pi}{4} + \frac{1}{3}\frac{\pi}{2} = \boxed{\frac{\pi}{4}}$

$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$

$p = 1 + \frac{1}{3} = \frac{4}{3} > 1$ convergent.

3. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x\sqrt[3]{x-1}}$

4. $\int_{-\infty}^{\infty} (2x^2 + x - 1)dx$ $\boxed{\text{diverges}}$

$x^2 = \frac{1}{\frac{1}{x^2}} = \frac{1}{x^{-2}}$ $\boxed{p=-2<1}$

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$p=2>1$, convergent.

5. $\int_0^{\infty} \frac{1}{(x+2)(x+3)}dx$

$\int \frac{1}{(x+2)(x+3)} dx$

Partial fractions: $\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$

$\int \frac{1}{(x+2)(x+3)} dx = \int \left[\frac{1}{x+2} - \frac{1}{x+3} \right] dx$

$= \ln|x+2| - \ln|x+3| + C = \ln\left|\frac{x+2}{x+3}\right| + C$

$\int_0^{\infty} \frac{1}{(x+2)(x+3)} dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x+2)(x+3)} dx = \lim_{t\to\infty} \left[\ln\left|\frac{x+2}{x+3}\right| \right]_0^t$

$= \lim_{t\to\infty} \left[\ln\left|\frac{t+2}{t+3}\right| - \ln\frac{2}{3} \right]$

$= \lim_{t\to\infty} \left[\underbrace{\ln\left|\frac{\cancel{t}(1+\cancel{\frac{2}{t}}^{0})}{\cancel{t}(1+\cancel{\frac{3}{t}}^{0})}\right|}_{\ln 1 = 0} - \ln\frac{2}{3} \right] = \boxed{-\ln\frac{2}{3}} = \boxed{\ln\frac{3}{2}}$

6. $\int_1^\infty e^x dx = \lim_{t\to\infty} \int_1^t e^x dx = \lim_{t\to\infty} e^x\Big]_1^t = \lim_{t\to\infty} (e^t - e) = \infty$

$\boxed{\text{divergent}}$

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$

7. $\int_{-\infty}^1 e^x dx = \lim_{t\to-\infty} \int_t^1 e^x dx = \lim_{t\to-\infty} e^x\Big]_t^1 = \lim_{t\to-\infty} (e - e^t) = \boxed{e}$ $\boxed{\text{convergent}}$

8. $\int_{-\infty}^\infty e^x dx = \int_{-\infty}^1 e^x dx + \int_1^\infty e^x dx = \text{convergent} + \text{divergent} = \boxed{\text{divergent}}$

9. $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$

$$\int \frac{dx}{x(\ln x)^2} = \left| \begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array} \right| = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$$\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \left[-\frac{1}{\ln x} \right]_e^t$$

$$= \lim_{t\to\infty} \left[-\overset{0}{\frac{1}{\ln t}} + \frac{1}{\cancel{\ln e}\,1} \right] = \boxed{1}$$

convergent.

$\boxed{\lim_{x\to\infty} \ln x = \infty}$

Definition of an improper integral of type 2 (discontinuous integrands)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\boxed{\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx}$$

if this limit exists (as a finite number)

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\boxed{\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx}$$

if this limit exists (as a finite number)

The improper integrals in (a) and (b) are called **convergent** if the limit exist and **divergent** if the limit does not exist.

(c) If f has discontinuity at c $(a < c < b)$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{s \to c^+} \int_s^b f(x)dx}$$

Example 3. For what values of p is the integral $\int_0^1 \frac{1}{x^p}dx$ convergent?

$\frac{1}{x^p}$ has discontinuity @ $x=0$.

1. $p>1 \Rightarrow 1-p<0$

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x^p}dx = \lim_{t \to 0^+} \int_t^1 x^{-p}dx$$

$$= \lim_{t \to 0^+} \frac{x^{-p+1}}{-p+1}\Big]_t^1 = \lim_{t \to 0^+} \left[\frac{1}{-p+1} - \frac{t^{-p+1}}{-p+1}\right]$$

$-p+1$ negative $\Rightarrow \frac{1}{0(-p+1)} \to \infty$

$$= \frac{1}{1-p} - \lim_{t \to 0^+} \frac{t^{-p+1}}{-p+1} = \infty,$$ divergent

2. $p=1$.

$$\int_0^1 \frac{1}{x}dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x}dx = \lim_{t \to 0^+} \left(\ln|x|\right]_t^1$$

$$\boxed{\lim_{x \to 0^+} (\ln x) = -\infty}$$

$$= \lim_{t \to 0^+} [\cancel{\ln 1}^{\,0} - \ln|t|] = -\lim_{t \to 0^+} (\ln|t|) = \infty$$, divergent

3. $p<1 \Rightarrow 1-p>0$

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x^p}dx = \lim_{t \to 0^+} \int_t x^{-p}dx$$

$$= \lim_{t \to 0^+} \frac{x^{-p+1}}{-p+1}\Big]_t^1 = \lim_{t \to 0^+} \left[\frac{1}{-p+1} - \frac{t^{-p+1}}{-p+1}\right]$$

$-p+1$ positive $\Rightarrow \frac{0}{-p+1} = 0$

$$= \frac{1}{1-p} - \lim_{t \to 0^+} \frac{t^{-p+1}}{-p+1} = \boxed{\frac{1}{1-p}},$$ convergent

$$\boxed{\int_0^1 \frac{1}{x^p}dx = \begin{cases} \frac{1}{1-p}, & \text{if } p<1 \quad (\text{convergent}) \\ \infty, & \text{if } p \geq 1 \quad (\text{divergent}) \end{cases}}$$

Example 4. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

$p = \frac{1}{2} < 1$ (convergent)

1. $\displaystyle\int_{-3}^{0} \frac{dx}{\sqrt{x+3}} = \lim_{t\to -3^+} \int_t^0 \frac{dx}{\sqrt{x+3}}$ $\left|\begin{array}{l} u = x+3 \\ du = dx \\ t \to t+3 \\ 0 \to 0+3 = 3 \end{array}\right|$ $= \lim_{t\to -3^+} \int_{t+3}^{3} \frac{du}{\sqrt{u}} = \lim_{t\to -3^+} \int_{t+3}^{3} u^{-1/2}\, du$

$$= \lim_{t\to -3^+} \left[2u^{1/2}\right]_{t+3}^{3} = 2\lim_{t\to -3^+} \left(\sqrt{3} - \sqrt{t+3}\right)$$

$$= 2\sqrt{3} - 2\cancel{\lim_{t\to -3^+}\sqrt{t+3}}^{\,0} = \boxed{2\sqrt{3}}$$

2. $\displaystyle\int_0^3 \frac{1}{x\sqrt{x}}\,dx$ $\qquad p = 1 + \frac{1}{2} = \frac{3}{2} > 1$ $\boxed{\text{divergent}}$

3. $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\,dx = \lim_{t\to \frac{\pi}{2}^-} \int_{\pi/4}^{t} \sec^2 x\,dx$

$\boxed{\sec x = \frac{1}{\cos x}, \quad \cos\frac{\pi}{2} = 0. \quad \lim_{x\to \frac{\pi}{2}} \sec x = \infty}$ $\boxed{\lim_{t\to \frac{\pi}{2}^-} \tan t = \infty}$

$$= \lim_{t\to \frac{\pi}{2}^-} \left[\tan x\right]_{\pi/4}^{t} = \lim_{t\to \frac{\pi}{2}^-} \left(\tan t - \cancel{\tan\frac{\pi}{4}}^{\,1}\right) = \infty \quad \boxed{\text{divergent}}$$

$$\lim_{x\to 0^+} (\ln x) = -\infty$$

4. $\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{array} \right|$

Integrate by parts.

$$= \lim_{t\to 0^+} \left(x\ln x \Big]_t^1 - \int_t^1 \overbrace{\frac{1}{x}x}^{1}\,dx \right) = \lim_{t\to 0^+} \left(1\cancel{\ln 1}^{0} - t\ln t - x \Big]_t^1 \right)$$

$$= \lim_{t\to 0^+} \left(-t\ln t - 1 + t \right) = -\lim_{t\to 0^+} \overbrace{(t\ln t)}^{0\cdot\infty} - 1 + \cancel{\lim_{t\to 0^+}(t)}^{0}$$

L'Hospital's Rule

$$= -\lim_{t\to 0^+} \frac{\ln t^{\,\infty}}{\frac{1}{t}\,\infty} - 1 = -\lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} - 1 = \lim_{t\to 0^+} \cancel{\frac{t^2}{t}}^{0} - 1 = \boxed{-1}$$

$\boxed{\text{convergent}}$

Comparison theorem Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example 5. Use the Comparison Theorem to determine whether $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent or divergent.

$-1 \leq \sin x \leq 1$

$$\frac{0}{x^2} \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$$

$$0 \leq \int_1^\infty \frac{\sin^2 x}{x^2} dx \leq \int_1^\infty \frac{1}{x^2} dx$$

$\int_1^\infty \frac{1}{x^2} dx$ — $p = 2 > 1$ convergent

By Comparison Theorem Part A, $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent.