Section 9.3 Arc length

Let's a curve C is defined by the equations

$$C: \boxed{x = x(t), \quad y = y(t), \quad a \le t \le b}$$

Assume that C is smooth ($x'(t)$ and $y'(t)$ are continuous and not simultaneously zero for $a < t < b$) Find the length of C when $a \le t \le b$.

Let P be a partition of $[a, b]$ into n subintervals of equal length Δt.

$$a = t_0 < t_1 < ... < t_n = b, \quad t_i = a + i\Delta t$$

Point $P_i(x(t_i), y(t_i))$ lies on C an the polygon with vertices P_0, P_1,...,P_n approximates C.

We define the length of C to be

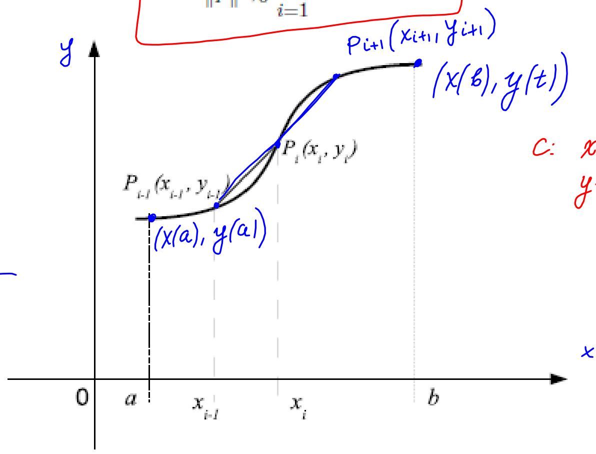
$$\boxed{L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} |P_{i-1}\ P_i|}$$

t: a —— b

n subintervals of equal length.

Take the points. $x_i = x(t_i)$

$y_i = y(t_i)$



C: $x = x(t)$, $y = y(t)$, $a \le t \le b$.

Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}\ P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

Since

$$x'(t_i) \approx \frac{\Delta x_i}{\Delta t}, \quad y'(t_i) \approx \frac{\Delta y_i}{\Delta t},$$

then

$$\Delta x_i = x'(t_i)\Delta t, \quad \Delta y_i = y'(t_i)\Delta t$$

Thus,

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2}\Delta t = \boxed{\int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt = L}$$

If the curve C is given by the equation

$$\boxed{y = y(x), \quad a \le x \le b, \quad \text{then} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx}$$

1

If the curve C is given by the equation

$$\boxed{x = x(y), \quad c \le y \le d, \quad \text{then} \quad L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy}$$

Example 1. Find the length of the curve

(a) $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 2$

$$L = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$x'(t) = 3 - 3t^2$

$y'(t) = 6t$

$$= \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2}\, dt = \int_0^2 \sqrt{(9 - 18t^2 + 9t^4) + 36t^2}\, dt$$

$$= \int_0^2 \sqrt{9(1 - 2t^2 + t^4 + 4t^2)}\, dt = 3\int_0^2 \sqrt{\underbrace{1 + 2t^2 + t^4}_{(t^2+1)^2}}\, dt$$

$$= 3\int_0^2 (t^2+1)\, dt = 3\left(\frac{t^3}{3} + t\right)\Big|_0^2 = \boxed{3\left(\frac{8}{3} + 2\right)}$$

(b) $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $1 \le x \le 2$

$$L = \int_1^2 \sqrt{1 + [y'(x)]^2}\, dx$$

$$y'(x) = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$L = \int_1^2 \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{4}\left(x^4 - 2\cancel{x^2}\frac{1}{\cancel{x^2}} + \frac{1}{x^4}\right)}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\, dx$$

$x^8 + 2x^4 + 1 = (x^4+1)^2$

$$= \int_1^2 \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}\, dx = \int_1^2 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\, dx$$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^2 = a^2 + 2ab + b^2$

$$= \int_1^2 \sqrt{\frac{(x^4+1)^2}{4x^4}}\, dx = \int_1^2 \frac{x^4+1}{2x^2}\, dx = \frac{1}{2}\int_1^2 (x^4+1)x^{-2}\, dx$$

$$= \frac{1}{2}\int_1^2 (x^2 + x^{-2})\, dx = \frac{1}{2}\left(\frac{x^3}{3} + \overbrace{\frac{x^{-1}}{-1}}^{-1/x}\right)\Big|_1^2$$

$$= \boxed{\frac{1}{2}\left(\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right)}$$

(c) $x = y^{3/2}$, $0 \le y \le 1$

$$L = \int_0^1 \sqrt{1 + [x'(y)]^2}\, dy$$

$$x'(y) = \frac{3}{2} y^{1/2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{3}{2} y^{1/2}\right)^2}\, dy = \frac{4}{9}\int_0^1 \frac{9}{4}\sqrt{1 + \frac{9}{4} y}\, dy =$$

$$\left| \begin{array}{l} u = 1 + \frac{9}{4} y \\ du = \frac{9}{4} dy \\ 0 \to 1 + \frac{9}{4}(0) = 1 \\ 1 \to 1 + \frac{9}{4}(1) = \frac{13}{4} \end{array} \right|$$

$$= \frac{4}{9} \int_1^{13/4} \sqrt{u}\, du = \frac{4}{9} \left. \frac{u^{3/2}}{3/2} \right]_1^{13/4}$$

$$= \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1 \right) = \boxed{\frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1 \right)}$$