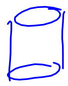
Section 9.4 **Area of a surface of revolution**

A surface of revolution is formed when a curve is rotated about a line.

Let's start with some simple surfaces.

The lateral surface area of a circular cylinder with base radius r and height h is

$$A = 2\pi r h$$

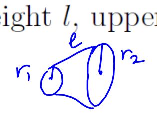


The lateral surface area of a circular cone with base radius r and slant height l is

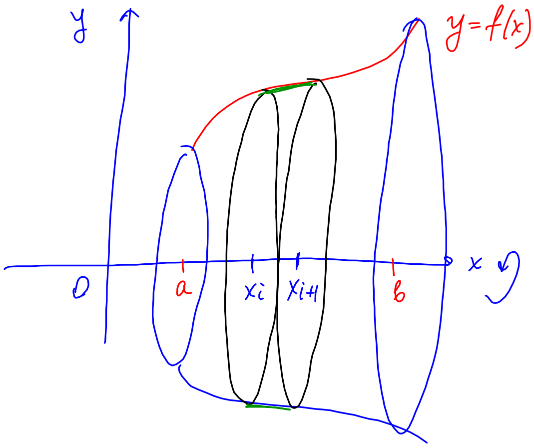
$$A = \pi r l$$

The lateral surface area of a band (frustum of a cone) with slant height l, upper radius r_1 and and lower radius r_2 is

$$\boxed{A = 2\pi r l} \text{ here } r = \frac{r_1 + r_2}{2}$$



Now we consider the surface which is obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis, $f(x) > 0$ for all x in $[a, b]$ and $f'(x)$ is continuous.



We take a partition P of $[a, b]$ by points $a = x_0 < x_1 < ... < x_n = b$, and let $y_i = f(x_i)$, so that the point $P_i(x_i, y_i)$ lies on the curve. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band with slant height $|P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$, its surface area is

$$S_i = 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

We know that

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + [f'(x_i^*)]^2} \Delta x_i$$

where $x_i^* \in [x_{i-1}, x_i]$. Since Δx_i is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ since f is continuous.

$$S_i \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \Delta x_i$$

Thus, the area of the complete surface is

$$S_X = 2\pi \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\sqrt{1+[f'(x_i^*)]^2}\Delta x_i =$$

$C: y = f(x), a \le x \le b$

$$2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}dx = \boxed{2\pi \int_a^b f(x)\sqrt{1+\left[\frac{df}{dx}\right]^2}dx = S.A._x}$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for the surface area is

$$\boxed{S_X = 2\pi \int_c^d y\sqrt{1+[g'(y)]^2}dy = 2\pi \int_c^d y\sqrt{1+\left[\frac{dx}{dy}\right]^2}dy}$$

Let's a curve C is defined by the equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

The area of the surface generated by rotating C about x-axis is

$$\boxed{S_X = 2\pi \int_a^b y(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}dt}$$

Example 1. Find the area of the surface obtained by rotating the curve about $\boxed{x\text{-axis}}$

(a) $y = \sqrt{x}$, $4 \le x \le 9$

$$S_x = 2\pi \int_4^9 y(x)\sqrt{1+(y'(x))^2}\,dx$$

$$y'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$= 2\pi \int_4^9 \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}dx = 2\pi \int_4^9 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$$

$$= 2\pi \int_4^9 \cancel{\sqrt{x}}\sqrt{\frac{4x+1}{4\cancel{x}}}\,dx = \pi\frac{1}{4}\int_4^9 4\sqrt{4x+1}\,dx$$

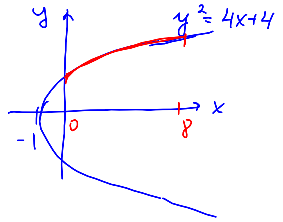
$u = 4x+1$
$du = 4dx$
$4 \to 4(4)+1 = 17$
$9 \to 4(9)+1 = 37$

$$= \frac{\pi}{4}\int_{17}^{37}\sqrt{u}\,du = \frac{\pi}{4}\frac{u^{3/2}}{3/2}\Big]_{17}^{37}$$

$$= \frac{\pi}{4}\frac{2}{3}\left(37^{3/2} - 17^{3/2}\right)$$

$$= \boxed{\frac{\pi}{6}\left(37\sqrt{37} - 17\sqrt{17}\right)}$$

(b) $y^2 = 4x + 4,\ 0 \le x \le 8$



Top half of the parabola.

solve for x: $y^2 = 4x+4$

$$x = \frac{y^2-4}{4} = \frac{1}{4}y^2 - 1, \quad x'(y) = \frac{y}{2}$$

corresponding limits for y.

$$y(0) = \sqrt{4} = 2$$
$$y(8) = \sqrt{32+4} = \sqrt{36} = 6$$
$$2 \le y \le 6$$

$$S_x = 2\pi \int_2^6 y\sqrt{1+(x'(y))^2}\,dy = 2\pi \int_2^6 y\sqrt{1+\left(\frac{y}{2}\right)^2}\,dy = \cancel{2}4\pi \int_2^6 \frac{y}{2}\sqrt{1+\frac{y^2}{4}}\,dy$$

$u = 1+\frac{y^2}{4}$
$du = \frac{y}{2}\,dy$
$2 \to 1+\frac{2^2}{4} = 2$
$6 \to 1+\frac{6^2}{4} = 10$

$$= 4\pi \int_2^{10} \sqrt{u}\,du = 4\pi \left.\frac{u^{3/2}}{3/2}\right]_2^{10} = 4\pi \frac{2}{3}\left(10^{3/2} - 2^{3/2}\right)$$

$$= \boxed{\frac{8\pi}{3}\left(10\sqrt{10} - 2\sqrt{2}\right)}$$

(c) $x(t) = a\cos^3 t,\ y(t) = a\sin^3 t,\ 0 \le t \le \pi/2$, a is a constant.

$$x'(t) = 3a\cos^2 t\,(\cos t)' = -3a\cos^2 t\sin t$$

$$y'(t) = 3a\sin^2 t\,(\sin t)' = 3a\sin^2 t\cos t$$

$$S_x = 2\pi \int_0^{\pi/2} y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t\sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t\sqrt{9a^2\sin^2 t\cos^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}\,dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t\ 3a\sin t\cos t\,dt$$

$$= 2\pi(3a^2)\int_0^{\pi/2} \sin^4 t\cos t\,dt$$

$u = \sin t$
$du = \cos t\,dt$
$0 \to \sin 0 = 0$
$\frac{\pi}{2} \to \sin\frac{\pi}{2} = 1$

$$= 6\pi a^2 \int_0^1 u^4\,du = 6\pi a^2 \left.\frac{u^5}{5}\right]_0^1 = \boxed{\frac{6\pi a^2}{5}}$$

For rotation about the y-axis, the surface area formulas are:

if the curve is given as $y = f(x)$, $a \le x \le b$, then the formula for the surface area is

$$S_Y = 2\pi \int_a^b x \sqrt{1 + \left[\frac{df}{dx}\right]^2}\, dx$$

if the curve is described as $x = g(y)$, $c \le y \le d$, then the formula for the surface area is

$$S_Y = 2\pi \int_c^d g(y) \sqrt{1 + \left[\frac{dg}{dy}\right]^2}\, dy$$

if the is defined by the equations $x = x(t)$,$y = y(t)$, $a \le t \le b$, then the area of the surface is

$$S_Y = 2\pi \int_a^b x(t) \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

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Example 2. Find the area of the surface obtained by rotating the curve about $\boxed{y\text{-axis}}$

(a) $x = \sqrt{2y - y^2}$, $0 \le y \le 1$

$$S_y = 2\pi \int_0^1 x(y) \sqrt{1 + (x'(y))^2}\, dy$$

$$x'(y) = \frac{1}{2}(2y - y^2)^{-1/2}(2y - y^2)' = \frac{1}{\cancel{2}}(2y-y^2)^{-1/2}(\cancel{2}-\cancel{2}y) = \frac{1-y}{\sqrt{2y-y^2}}$$

$$S_y = 2\pi \int_0^1 \sqrt{2y-y^2} \sqrt{1 + \left(\frac{1-y}{\sqrt{2y-y^2}}\right)^2}\, dy = 2\pi \int_0^1 \sqrt{2y-y^2} \sqrt{1 + \frac{(1-y)^2}{2y-y^2}}\, dy$$

$$= 2\pi \int_0^1 \cancel{\sqrt{2y-y^2}} \sqrt{\frac{\cancel{2y}-\cancel{y^2} + 1 - \cancel{2y} + \cancel{y^2}}{\cancel{2y-y^2}}}\, dy = 2\pi \int_0^1 1\, dy = 2\pi\, y \Big]_0^1 = \boxed{2\pi}$$

(b) $y = 1 - x^2$, $0 \le x \le 1$ $\quad y'(x) = -2x$

$$S_y = 2\pi \int_0^1 x \sqrt{1 + (y'(x))^2}\, dx = 2\pi \int_0^1 x \sqrt{1 + (-2x)^2}\, dx$$

$$= \frac{2\pi}{8} \int_0^1 8x \sqrt{1+4x^2}\, dx$$

$u = 1 + 4x^2$
$du = 8x\, dx$
$0 \to 1 + 4(0) = 1$
$1 \to 1 + 4(1) = 5$

$\boxed{a^{3/2} = a\sqrt{a}}$

$$= \frac{\pi}{4} \int_1^5 \sqrt{u}\, du = \frac{\pi}{4} \frac{u^{3/2}}{3/2}\Big]_1^5 = \frac{\pi}{4} \cdot \frac{2}{3}\left(5^{3/2} - 1^{3/2}\right)$$

$$= \boxed{\frac{\pi}{6}\left(5\sqrt{5} - 1\right)}$$

(c) $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 1$

$$S_y = 2\pi \int_0^1 x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$\boxed{(e^t)^2 = e^{2t}}$$

$$x'(t) = e^t - 1, \quad y'(t) = 4e^{t/2}\left(\tfrac{1}{2}\right) = 2e^{t/2}$$

$$= 2\pi\int_0^1 (e^t - t)\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\,dt$$

$$= 2\pi\int_0^1 (e^t-t)\sqrt{e^{2t} \underline{-2e^t} + 1 + \underline{4e^t}}\,dt$$

$$= 2\pi\int_0^1 (e^t-t)\sqrt{e^{2t}+2e^t+1}\,dt$$

$$= 2\pi\int_0^1 (e^t-t)\sqrt{(e^t+1)^2}\,dt$$

$$= 2\pi\int_0^1 (e^t-t)(e^t+1)\,dt = 2\pi\int_0^1 (e^{2t}+e^t-te^t-t)\,dt$$

$$\left| \begin{array}{l} u=t,\ v'=e^t \\ u'=1,\ v=e^t \end{array}\right|$$

$$= 2\pi\int_0^1 e^{2t}\,dt + 2\pi\int_0^1 e^t\,dt - 2\pi\int_0^1 t\,dt - 2\pi\int_0^1 te^t\,dt =$$

$$= \frac{2\pi}{2}e^{2t}\Big]_0^1 + 2\pi e^t\Big]_0^1 - 2\pi\frac{t^2}{2}\Big]_0^1 - 2\pi\left(te^t\Big]_0^1 - \int_0^1 e^t\,dt\right)$$

$$= \pi(e^2-1) + 2\pi(e-1) - \pi - 2\pi\left(e - e^t\Big]_0^1\right)$$

$$= \pi e^2 - \pi + \cancel{2\pi e} - 2\pi - \pi - \cancel{2\pi e} + \underbrace{2\pi(e-1)}_{2\pi e - 2\pi}$$

$$= \boxed{\pi e^2 + 2\pi e - 6\pi}$$