Section 7.3 **Volumes by cylindrical shells**

Lets find the volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h (see Fig.1).

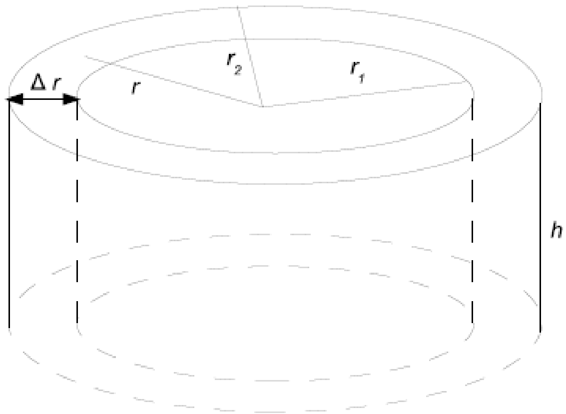


Fig.1

V can be calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi h(r_2^2 - r_1^2) = \boxed{2\pi h\frac{r_2 + r_1}{2}(r_2 - r_1) = V}$$

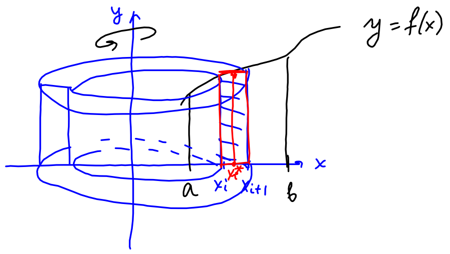
Let $\Delta r = r_2 - r_1$, $r = (r_2 + r_1)/2$, then the volume of a cylindrical shell is

$$V = 2\pi r h \Delta r$$

$V =$[circumference][height][thickness]

$\boxed{V = 2\pi\text{[average radius][height][thickness]}}$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.



approximate the part of the solid between $x = x_i$ and x_{i+1} by a cylindrical shell of heigh $f(x_i^*)$

$x_i < x_i^* < x_{i+1}$

Volume of this shell is $V_i = 2\pi \overbrace{f(x_i^*)}^{\text{height}} \underbrace{x_i^*}_{\text{average radius}} \Delta x_i$

Let P be a partition of $[a, b]$ by points x_i such that $a = x_0 < x_1 < \ldots < x_n = b$ and let x_i^* be the midpoint of $[x_{i-1}, x_i]$, that is $x_i^* = (x_{i-1} + x_i)/2$. If the rectangle with base $[x_{i-1}, x_i]$

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and height $f(x_i^*)$ is rotated about the y-axis, then the result is a cylindrical shell with average raduis x_i^*, height $f(x_i^*)$, and thikness $\Delta x_i = x_i - x_{i-1}$, so its volume is $V_i = 2\pi x_i^* f(x_i^*)\Delta x_i$.

The approximation to the volume V of S is $V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i$. This approximation appears to become better and better as $\|P\| \to 0$.

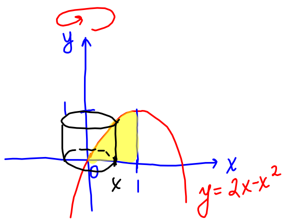
Thus, the volume of S is

radius of a shell

$$V_Y = \lim_{\|P\| \to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x_i = \boxed{2\pi \int_a^b x f(x)dx = V_y}$$

height of a shell

Example 1. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 0$, $0 \leq x \leq 1$ about the y-axis.



$0 \leq x \leq 1$

Pick $0 \leq x \leq 1$, do a vertical line through x. rotate the line about the y-axis.

[radius] = x

[height] = $2x - x^2$

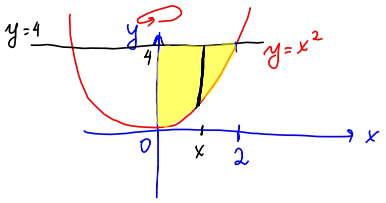
$$V = 2\pi \int_0^1 x(2x - x^2)\,dx = 2\pi \int_0^1 (2x^2 - x^3)\,dx$$

$$= 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \boxed{2\pi\left(\frac{2}{3} - \frac{1}{4}\right)}$$

The volume of the solid generated by rotating about the y-axis the region between the curves $y = f(x)$ and $y = g(x)$ from a to b $[f(x) \geq g(x)$ and $0 \leq a < b]$ is

$$\boxed{V_Y = 2\pi \int_a^b \underbrace{x}_{\text{radius}} \overbrace{[f(x) - g(x)]}^{\text{height}} dx}$$

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, $x = 0$ about the y-axis, $x > 0$.



$0 \leq x \leq 2$

$[\text{radius}] = x$

$[\text{height}] = [\text{top}] - [\text{bottom}] = 4 - x^2$

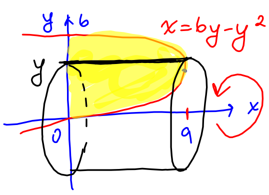
$V = 2\pi \int_0^2 x(4-x^2)\,dx = 2\pi \int_0^2 (4x - x^3)\,dx$

$= 2\pi \left(\frac{4x^2}{2} - \frac{x^4}{4} \right)\Big|_0^2 = 2\pi \left(2(4) - \frac{16}{4} \right) = \boxed{8\pi}$

The method of cylindrical shells also allows us to compute volumes of revolution about the x-axis. If we interchange the roles of x and y in the formula for the volume, then the volume of the solid generated by rotating the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the x-axis, is

$$\boxed{V_X = 2\pi \int_c^d y g(y) dy}$$

Example 3. Find the volume of the solid obtained by rotating the region bounded by $y^2 - 6y + x = 0$, $x = 0$ about the x-axis.



$x = 6y - y^2$, $0 \le y \le 6$

Pick any $0 \le y \le 6$, do a horizontal line through y, rotate the line about the x-axis.

[radius] = y

[height] = $6y - y^2$

$V = 2\pi \int_0^6 y(6y - y^2)\,dy = 2\pi \int_0^6 (6y^2 - y^3)\,dy$

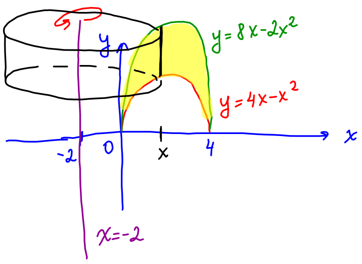
$= 2\pi \left(\frac{6y^3}{3} - \frac{y^4}{4} \right)_0^6 = 2\pi \left(2(6^3) - \frac{6^4}{4} \right)$

$= \boxed{2\pi \left(2(216) - \frac{1296}{4} \right)}$

The volume of the solid generated by rotating the region bounded by $x = g_1(y)$, $x = g_2(y)$, $y = c$, and $y = d$, about the x-axis, assuming that $g_2(y) \geq g_1(y)$ for all $c \leq x \leq d$, is

$$\boxed{V_X = 2\pi \int_c^d y[g_2(y) - g_1(y)]dy}$$

Example 4. Find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$, $y = 8x - 2x^2$ about $x = -2$.



$0 \leq x \leq 4$

$[\text{radius}] = 2 + x$

$[\text{height}] = 8x - 2x^2 - (4x - x^2)$
$= 4x - x^2$

$$V = 2\pi \int_0^4 (2+x)(4x - x^2)\, dx$$

$$= 2\pi \int_0^4 (8x - 2x^2 + 4x^2 - x^3)\, dx = 2\pi \int_0^4 (8x + 2x^2 - x^3)\, dx$$

$$= 2\pi \left(\frac{8x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} \right)\Big|_0^4 = \boxed{2\pi \left(4(16) + \frac{2}{3}(64) - \frac{256}{4} \right)}$$