MATH 152, 525-530, 534-536, Spring 2013, Sample problems for Test 2

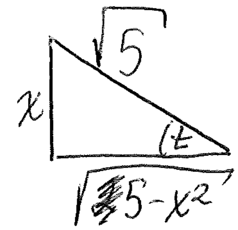
1. Evaluate the integral

(a) $\int \frac{x^2}{\sqrt{5-x^2}}dx = \left| \begin{array}{l} x=\sqrt{5}\sin t \\ dx=\sqrt{5}\cos t\, dt \\ \sqrt{5-x^2}=\sqrt{5}\cos t \end{array} \right|$

$$= \int \frac{5\sin^2 t}{\sqrt{5}\cos t}\sqrt{5}\cos t\,dt = 5\int \sin^2 t\,dt$$

$$= \frac{5}{2}\int (1-\cos 2t)\,dt = \frac{5}{2}\left(t - \frac{1}{2}\sin 2t\right) + t$$

$$x=\sqrt{5}\sin t \longrightarrow t = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$$



$$\sin 2t = 2\sin t\cos t$$

$$\sin t = \frac{x}{\sqrt{5}}$$

$$\cos t = \frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$\sin 2t = 2\,\frac{x}{\sqrt{5}}\,\frac{\sqrt{5-x^2}}{\sqrt{5}} = \frac{2}{5}x\sqrt{5-x^2}$$

$$\int \frac{x^2}{\sqrt{5-x^2}}dx = \frac{5}{2}\left(\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) - \frac{2}{5}x\sqrt{5-x^2}\right) + C$$

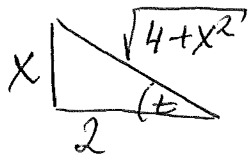
$$= \boxed{\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) - x\sqrt{5-x^2} + C}$$

(b) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$ $\quad\left|\begin{array}{l} x = 2\tan t \\ dx = 2\sec^2 t\,dt \\ \sqrt{x^2+4} = 2\sec t \end{array}\right|$

$$= \int \frac{8\tan^3 t}{2\sec t}\, 2\sec^2 t\,dt = 8\int \tan^3 t \sec t\,dt \quad \left|\begin{array}{l} u = \sec t \\ du = \sec t \tan t\,dt \\ \tan^2 t = \\ \sec^2 t - 1 \\ = u^2 - 1 \end{array}\right.$$

$$= 8\int (u^2-1)\,du = 8\left(\frac{u^3}{3} - u\right) + C \overset{*}{=} 8\left(\frac{\sec^3 t}{3} - \sec t\right) + C$$

$x = 2\tan t \longrightarrow \tan t = \dfrac{x}{2}$



$\cos t = \dfrac{2}{\sqrt{4+x^2}}$

$\sec t = \dfrac{1}{\cos t} = \dfrac{\sqrt{4+x^2}}{2}$

$$\overset{*}{=} 8\left(\frac{1}{8\cdot 3}(4+x^2)^{3/2} - \frac{\sqrt{4+x^2}}{2}\right) + C$$

$$= \boxed{\frac{1}{3}(4+x^2)^{3/2} - 4(4+x^2)^{1/2} + C}$$

OR

$$\frac{1}{2}\int \frac{2x^3}{\sqrt{x^2+4}}\,dx = \left|\begin{array}{l} x^2+4 = u \\ du = 2x\,dx \\ x^2 = u - 4 \end{array}\right| = \frac{1}{2}\int \frac{u-4}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int \left(\sqrt{u} - 4u^{-1/2}\right)du = \frac{1}{2}\left(\frac{2u^{3/2}}{3} - \frac{4u^{1/2}}{1/2}\right) + C$$

$$= \frac{1}{2}\left(\frac{2(x^2+4)^{3/2}}{3} - 8(x^2+4)^{1/2}\right) + C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4(x^2+4)^{1/2} + C}$$

(c) $\int \frac{dx}{\sqrt{x^2+4x-5}}$

complete the square:

$$x^2+4x-5 = \underbrace{(x^2+4x+4)}_{(x+2)^2} - 4 - 5$$

$$= (x+2)^2 - 9$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-9}}$$

$x+2 = 3\sec t$

$dx = 3\sec t \tan t \, dt$

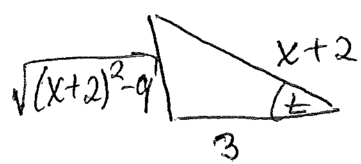
$\sqrt{(x+2)^2-9} = \sqrt{9\sec^2 t - 9}$

$= 3\tan t$

$$= \int \frac{3\sec t \tan t \, dt}{3\tan t} = \int \sec t \, dt$$

$$= \ln|\sec t + \tan t| + C$$

$\cos t = \frac{3}{x+2}$



$x+2 = 3\sec t \longrightarrow \sec t = \frac{x+2}{3}$

$\tan t = \frac{\sqrt{(x+2)^2-9}}{3}$

$$= \boxed{\ln\left| \frac{x+2}{3} + \frac{\sqrt{(x+2)^2-9}}{3} \right| + C}$$

(d) $\int \frac{dx}{x^2(x^2+1)}$

Partial fraction decomposition:

$$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$$

$$1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$1 = x^3(A+C) + x^2(B+D) + Ax + B$$

x^3: $A+C=0 \longrightarrow C=0$

x^2: $B+D=0 \longrightarrow D=-1$

x: $A=0$

: $1=B$

$$\frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$$

$$\int \frac{dx}{x^2(x^2+1)} = \int \frac{dx}{x^2} - \int \frac{dx}{x^2+1}$$

$$= \boxed{-\frac{1}{x} - \tan^{-1}x + C}$$

(e) $\int \frac{x^2+3x-1}{x-1}dx$

Improper fraction.
Divide x^2+3x-1 ~~by~~ into $x-1$ by long divisions:

$$\begin{array}{r} x+4 \\ x-1 \,\big)\,\overline{x^2+3x-1} \\ -x^2-x \\ \hline 4x-1 \\ -4x-4 \\ \hline 3 \end{array} \longrightarrow \frac{x^2+3x-1}{x-1} = x+4+\frac{3}{x-1}$$

$$\int \frac{x^2+3x-1}{x-1}dx = \int\left(x+4+\frac{3}{x-1}\right)dx$$

$$= \boxed{\frac{x^2}{2}+4x+3\ln|x-1|+C}$$

OR

$$\left|\begin{array}{ll} x-1=u & x=u+1 \\ dx=du & \end{array}\right|$$

$$\int \frac{x^2+3x-1}{x-1}dx = \int \frac{(u+1)^2+3(u+1)-1}{u}du$$

$$= \int \frac{u^2+2u+1+3u+3-1}{u}du = \int \frac{u^2+5u+3}{u}du$$

$$= \int\left[u+5+\frac{3}{u}\right]du = \frac{u^2}{2}+5u+3\ln|u|+C$$

$$= \boxed{\frac{(x-1)^2}{2}+5(x-1)+3\ln|x-1|+C}$$

(f) $\int_0^{\infty} \frac{dx}{(x+2)(x+3)}$

$$\frac{1}{(x+2)(x+3)} < \frac{1}{x^2}$$

$\int_a^{\infty} \frac{1}{x^2}\,dx$ is convergent, therefore

$\int_0^{\infty} \frac{dx}{(x+2)(x+3)}$ converges.

Partial fractions:

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$= \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$$

$$1 = A(x+3) + B(x+2)$$

$x = -3$: $\quad 1 = -B \rightarrow B = -1$

$x = -2$: $\quad 1 = A \rightarrow A = 1$

$$\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\int_0^{\infty} \frac{dx}{(x+2)(x+3)} = \lim_{t\to\infty} \int_0^t \left[\frac{1}{x+2} - \frac{1}{x+3}\right] dx$$

$$= \lim_{t\to\infty} \left[\ln|x+2| - \ln|x+3|\right]_0^t$$

$$= \lim_{t\to\infty} \left[\ln|t+2| - \ln|t+3| - \ln 2 + \ln 3\right]$$

$$= \cancel{\lim_{t\to\infty} \left[\ln\left|\frac{t+2}{t+3}\right|\right]}^{\,0} - \ln 2 + \ln 3 = \boxed{\ln 3 - \ln 2}$$

(g) $\displaystyle\int_{-\infty}^{1} \frac{dx}{(2x-3)^2}$

$$\cancel{\int_{-\infty} \frac{dx}{d}} \qquad \frac{1}{(2x-3)^2} < \frac{1}{x^2}$$

$\displaystyle\int_{-\infty}^{1} \frac{1}{x^2}\,dx$ is convergent,

therefore $\displaystyle\int_{-\infty}^{1} \frac{dx}{(2x-3)^2}$ converges.

$$\int_{-\infty}^{1} \frac{dx}{(2x-3)^2} = \lim_{t\to-\infty} \int_{t}^{1} \frac{dx}{(2x-3)^2}$$

$$= \lim_{t\to-\infty} \left[-\frac{1}{2} \frac{1}{2x-3} \Big|_t^1 \right]$$

$$= -\frac{1}{2}\frac{1}{2-3} + \frac{1}{2}\cancel{\lim_{t\to-\infty} \frac{1}{2t-3}} \to 0$$

$$= \boxed{\frac{1}{2}}$$

(h) $\displaystyle\int_4^5 \frac{dx}{(5-x)^{2/5}} = \lim_{t\to 5^-} \int_4^t \frac{dx}{(5-x)^{2/5}}$

$$= -\lim_{t\to 5^-} \left.\frac{(5-x)^{-2/5+1}}{-2/5+1}\right|_4^t$$

$$= -\lim_{t\to 5^-} \frac{5(5-t)^{3/5}}{3} + \frac{5}{3}\, 1^{3/5}$$

$$= -\frac{5}{3}\cancelto{0}{\lim_{t\to 5^-} (5-t)^{3/5}} + \frac{5}{3}$$

$$= \boxed{\frac{5}{3}}$$

2. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3+x-1}{(x^2-1)(x+1)(x^2+1)^2}.$$

Do not determine the numerical values for the coefficients.

$$\frac{x^3+x-1}{(x^2-1)(x+1)(x^2+1)^2}$$

$$=\frac{x^3+x-1}{(x-1)(x+1)(x+1)(x^2+1)^2}$$

$$=\frac{x^3+x-1}{(x-1)(x+1)^2(x^2+1)^2}$$

$$=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}$$