3. Use the Comparison Theorem to determine which of the following integrals is convergent.

(a) $\displaystyle\int_3^\infty \frac{3+\sin x}{x}dx$

(b) $\displaystyle\int_1^\infty \frac{2+\cos x}{x^2}dx$

<u>Comparison Thm.</u> Given $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$ such that $f(x) \le g(x)$ on $[a,\infty)$

Then:

(a) if $\int_a^\infty g(x)\,dx$ convergent, then $\int_a^\infty f(x)\,dx$ is convergent.

(b) if $\int_a^\infty f(x)\,dx$ is divergent, then $\int_a^\infty g(x)$ is divergent

(a) $\displaystyle\int_3^\infty \frac{3+\sin x}{x}\,dx$

$$-1 \le \sin x \le 1$$
$$3+(-1) \le 3+\sin x \le 1+3$$
$$\frac{2}{x} \le \frac{3+\sin x}{x} \le \frac{4}{x}$$
$$\underbrace{\int_3^\infty \frac{2}{x}\,dx}_{\text{divergent}} \le \int_3^\infty \frac{3+\sin x}{x}\,dx \le \underbrace{\int_3^\infty \frac{4}{x}\,dx}_{\text{divergent}}$$

$\boxed{\int_3^\infty \frac{3+\sin x}{x}\,dx \text{ diverges by comparison with } \int_3^\infty \frac{2}{x}\,dx}$

(b) $\displaystyle\int_1^\infty \frac{2+\cos x}{x^2}\,dx$

$$-1 \le \cos x \le 1$$
$$2-1 \le 2+\cos x \le 2+1$$
$$\frac{1}{x^2} \le \frac{2+\cos x}{x^2} \le \frac{3}{x^2}$$
$$\underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{convergent}} \le \int_1^\infty \frac{2+\cos x}{x^2}\,dx \le \underbrace{\int_1^\infty \frac{3}{x^2}\,dx}_{\text{convergent}}$$

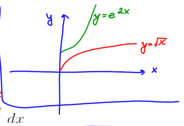
$\boxed{\int_1^\infty \frac{2+\cos x}{x^2}\,dx \text{ is convergent by comparison with } \int_1^\infty \frac{3}{x^2}\,dx.}$

$$\int_2^\infty \frac{\boxed{4x^2}+2x-1}{\boxed{3x^3}-4x^2+4x+8}\,dx$$

compare with $\displaystyle\int_2^\infty \frac{4x^2}{3x^3}\,dx = \int_2^\infty \frac{4}{3x}\,dx$ $(p=1 \Rightarrow \text{divergent})$

$\boxed{\text{divergent by comparison with } \int_2^\infty \frac{4}{3x}\,dx.}$

11. The integral $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+e^{2x}} \le \int_1^\infty \frac{dx}{e^{2x}} = \underbrace{\int_1^\infty e^{-2x}\,dx}_{\text{convergent}}$



(a) converges to 0.

(b) diverges by comparison to $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$.

(c) converges by comparison to $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$.

(d) converges by comparison to $\displaystyle\int_1^\infty e^{-2x}\,dx$.

(e) diverges by comparison to $\displaystyle\int_1^\infty e^{-2x}\,dx$.

$$\int_1^\infty e^{-2x}\,dx$$
$$= \lim_{t\to\infty} \int_1^t e^{-2x}\,dx$$
$$= \lim_{t\to\infty} \left(-\frac{1}{2}e^{-2x}\right)\Big|_1^t$$
$$= \lim_{t\to\infty} \left(\cancel{-\frac{1}{2}e^{-2t}} + \frac{1}{2}e^{-2}\right)$$
$$= \frac{1}{2}e^{-2}$$
is convergent.

6. Which of the following statements is true for the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{\sqrt{1+4n^2}}$?

~~I. It converges by the Divergence Test.~~ false

~~II. It converges to~~ $\frac{3}{2}$.

III. It diverges.

(a) Only I is true.

(b) Only II is true.

(c) Only III is true.

(d) Only I and II are true.

(e) All three statements I, II, and III, are false.

$a_n = \frac{3n}{\sqrt{1+4n^2}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\boxed{3n}}{\sqrt{1+\boxed{4n^2}}}$

$= \lim_{n\to\infty} \frac{3n}{\sqrt{4n^2}} = \lim_{n\to\infty} \frac{3n}{2n} = \frac{3}{2} \neq 0$

$\boxed{\sum_{n=1}^{\infty} \frac{3n}{\sqrt{1+4n^2}} \text{ is divergent by the Test for Divergence}}$

9. Given the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$ by $s_n = \dfrac{n}{2n+1}$, find a_4.

(a) $\dfrac{4}{9}$

(b) $\dfrac{1}{35}$

(c) $\dfrac{1}{99}$

(d) $\dfrac{3}{7}$

(e) $\dfrac{1}{63}$

$S_n = \dfrac{n}{2n+1}$, $a_n = S_n - S_{n-1}$

$a_4 = S_4 - S_3$

$= \underbrace{\dfrac{4}{2(4)+1}}_{S_4} - \underbrace{\dfrac{3}{2(3)+1}}_{S_3}$

$= \dfrac{4}{9} - \dfrac{3}{7} = \dfrac{1}{63}$

2. Which of the series below is convergent?

(I) $\sum_{n=1}^{\infty} \frac{6n+3}{n+1}$ (II) $\sum_{n=1}^{\infty} \frac{6n+3}{n(n+1)}$ (III) $\sum_{n=1}^{\infty} \frac{6n+3}{n^2(n+1)}$

(a) (III) only
(b) (II) and (III) only
(c) None of the other answers is correct.
(d) All 3 are convergent.
(e) All 3 are divergent.

(I) $\sum_{n=1}^{\infty} \frac{6n+3}{n+1}$ $a_n = \frac{6n+3}{n+1}$

$\lim_{n\to\infty} a_n = 6 \neq 0$

divergent by the Test for Divergence.

(II) $\sum_{n=1}^{\infty} \frac{6n+3}{n(n+1)}$ $a_n = \frac{6n+3}{n(n+1)}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6n+3}{n(n+1)} = \lim_{n\to\infty} \frac{6n}{n^2}$

$= \lim_{n\to\infty} \frac{6}{n} = 0$

$\sum_{n=1}^{\infty} \frac{\boxed{6n}+3}{\boxed{n(n}+1)}$ | $\sum_{n=1}^{\infty} \frac{6n}{n^2} = \sum_{n=1}^{\infty} \frac{6}{n}$

harmonic series (divergent)

(III) $\sum_{n=1}^{\infty} \frac{6n+3}{n^2(n+1)}$ — is convergent.

$\sum_{n=1}^{\infty} \frac{6n}{n^3} = \sum_{n=1}^{\infty} \frac{6}{n^2}$ convergent. $(p=2>1)$

8. Assume that the sequence $\{a_n\}$ is bounded, increasing and given by

$$a_1 = 3 \qquad \text{and} \qquad a_{n+1} = 6 - \frac{8}{a_n}$$

for all positive integers n. Determine if the sequence is convergent or divergent.

(a) Divergent

(b) Convergent to 2

(c) Convergent to 4

(d) Convergent to 6

(e) Convergent to 8

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$

$3 < L$

$L = 6 - \frac{8}{L}$

$L^2 - 6L + 8 = 0$

$(L-4)(L-2) = 0$

$L_1 = \cancel{2}, \quad L_2 = 4.$

3. Use the Comparison Theorem to determine which of the following integrals is convergent.

(a) $\displaystyle\int_3^\infty \frac{3+\sin x}{x}\,dx$

(b) $\displaystyle\int_1^\infty \frac{2+\cos x}{x^2}\,dx$

$0<\frac{2}{n}\le 2 \Rightarrow -2\le -\frac{2}{n}<0$

$1-2\le 1-\frac{2}{n}<1$

(a) $\left(1-\frac{2}{n}\right)' = \frac{2}{n^2}>0$ increasing

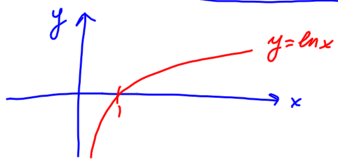
$-1\le 1-\frac{2}{n}<1$ bounded

(b) $a_n=\ln n$

increasing

$\lim\limits_{n\to\infty}\ln n=\infty$

not bounded



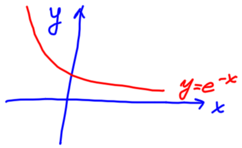
(c) $a_n=\sin(2\pi n)$

$-1\le \sin(2\pi n)\le 1$ bounded

not monotonic.

(d) $a_n=e^{-n}$

decreasing



Find the sum of the series

$$10-4+\frac{8}{5}-\frac{16}{25}+\dots$$

$$=10-4\left(1-\frac{2}{5}+\frac{4}{25}-\dots\right)$$

$$=10-4\underbrace{\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n}_{\text{geometric, } a=1,\ r=-\frac{2}{5}}$$

$$=10-4\cdot\frac{1}{1-\left(-\frac{2}{5}\right)}=10-4\cdot\frac{1}{\frac{7}{5}}=10-\frac{20}{7}=\boxed{\frac{50}{7}}$$

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(e^{\frac{5}{n}}-e^{\frac{5}{n+1}}\right)$$

Partial sums:

$$S_1=e^{\frac{5}{1}}-e^{\frac{5}{2}}=a_1$$

$$S_2=a_1+a_2=\underbrace{e^{\frac{5}{1}}-\cancel{e^{\frac{5}{2}}}}_{a_1}+\underbrace{\cancel{e^{\frac{5}{2}}}-e^{\frac{5}{3}}}_{a_2}$$

$$=e^5-e^{\frac{5}{3}}$$

$$S_3=S_2+a_3=e^5-\cancel{e^{\frac{5}{3}}}+\cancel{e^{\frac{5}{3}}}-e^{\frac{5}{4}}$$

$$=e^5-e^{\frac{5}{4}}$$

$$S_n=e^5-e^{\frac{5}{n+1}}$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(e^5-e^{\cancel{\frac{5}{n+1}}\to 0}\right)=e^5-e^0=\boxed{e^5-1}$$

Find the limit:

(a) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^3}$

(b) $\lim\limits_{n\to\infty} \frac{(-3)^n n^3}{n^3+1}$

(c) $\lim\limits_{n\to\infty} \frac{(-1)^n (n^2+4)}{\sqrt{n^4+1}}$

Theorem: if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

(a) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^3}$ $\quad a_n = \frac{(-1)^n}{n^3}$, $|a_n| = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{1}{n^3} = 0 \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n^3} = 0$$

(b) $\lim\limits_{n\to\infty} \frac{(-3)^n \cancel{n^3}}{\cancel{n^3}+1}$ $\boxed{DNE}$

if n is $\boxed{\text{even}}$, then $a_n = \frac{3^n(n^3)}{n^3+1}$

and $\lim\limits_{n\to\infty} \frac{3^n n^3}{n^3+1} = \lim\limits_{n\to\infty} 3^n = \infty$

if n is odd, then $a_n = -\frac{3^n n^3}{n^3+1}$

and $\lim\limits_{n\to\infty} \left(-\frac{3^n n^3}{n^3+1}\right) = -\infty$

(c) $\lim\limits_{n\to\infty} \frac{(-1)^n (n^2+4)}{\sqrt{n^4+1}}$ $\boxed{DNE}$

if n is even, then $a_n = \frac{n^2+4}{\sqrt{n^4+1}}$

$\lim\limits_{n\to\infty} a_n = 1$

if n is odd, then $a_n = -\frac{n^2+4}{\sqrt{n^4+1}}$

$\lim\limits_{n\to\infty} a_n = -1$

Find $\lim_{n\to\infty} a_n$

2. Which of the series below is convergent?

(I) $\sum_{n=1}^{\infty} \frac{6n+3}{n+1}$ (II) $\sum_{n=1}^{\infty} \frac{6n+3}{n(n+1)}$ (III) $\sum_{n=1}^{\infty} \frac{6n+3}{n^2(n+1)}$

(a) (III) only
(b) (II) and (III) only
(c) None of the other answers is correct.
(d) All 3 are convergent.
(e) All 3 are divergent.

$\boxed{\text{L'Hospital's Rule.}}$

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n}$$

$$= \lim_{n\to\infty} \frac{2\ln n \, \frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2\ln n}{n}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n}}{1} = \lim_{n\to\infty} \frac{2}{n} = \boxed{0}$$

$$\lim_{n\to\infty} n^2 e^{-3n} \quad |0\cdot\infty|$$

$$= \lim_{n\to\infty} \frac{n^2}{e^{3n}} = \lim_{n\to\infty} \frac{2n}{3e^{3n}} = \lim_{n\to\infty} \frac{2}{9e^{3n}} = \boxed{0}$$