20. Find the angle between the vectors $\vec{a} = \vec{\imath} + \vec{\jmath} + 2\vec{k}$ and $\vec{b} = 2\vec{\jmath} - 3\vec{k}$.

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{1(0)+1(2)+2(-3)}{\sqrt{1^2+1^2+2^2}\cdot\sqrt{2^2+3^2}} = \frac{-1}{\sqrt{6}\cdot\sqrt{13}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{6}\sqrt{13}}\right)$$

or $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\cdot|\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 2 \\ 0 & 2 & -3 \end{vmatrix} = \vec{\imath}\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= \vec{\imath}(-3-4) - \vec{\jmath}(-3) + 2\vec{k}$$

$$= -7\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$$

$$\sin\theta = \frac{\sqrt{(-7)^2+3^2+2^2}}{\sqrt{6}\cdot\sqrt{13}} = \frac{\sqrt{49+9+4}}{\sqrt{6}\sqrt{13}} = \frac{\sqrt{62}}{\sqrt{78}} = \sqrt{\frac{62}{78}}, \quad \theta = \arcsin\sqrt{\frac{62}{78}}$$

21. Find the directional cosines for the vector $\vec{a} = -2\vec{\imath} + 3\vec{\jmath} + \vec{k}$.

$$\frac{\vec{a}}{|\vec{a}|} = \frac{\langle -2, 3, 1 \rangle}{\sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{\langle -2, 3, 1 \rangle}{\sqrt{14}} = \langle \underbrace{-\frac{2}{\sqrt{14}}}_{\cos\alpha}, \underbrace{\frac{3}{\sqrt{14}}}_{\cos\beta}, \underbrace{\frac{1}{\sqrt{14}}}_{\cos\gamma} \rangle$$

22. Find the scalar and the vector projections of the vector $\vec{a} = < 2, -3, 1 >$ onto the vector $\vec{b} = < 1, 6, -2 >$.

scalar $\quad \text{comp}_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{2(1)+(-3)(6)+1(-2)}{\sqrt{1^2+6^2+(-2)^2}} = \dfrac{2-18-2}{\sqrt{1+36+4}} = \boxed{-\dfrac{18}{\sqrt{41}}}$

vector $\quad \text{proj}_{\vec{b}}\vec{a} = \left(\text{comp}_{\vec{b}}\vec{a}\right)\dfrac{\vec{b}}{|\vec{b}|} = -\dfrac{18}{\sqrt{41}}\,\dfrac{<1,6,-2>}{\sqrt{41}} = -\dfrac{18}{41}<1,6,-2>$

$= \boxed{< -\dfrac{18}{41}, \dfrac{-108}{41}, \dfrac{36}{41} >}$

23. Given vectors $\vec{a} =< -2, 3, 4 >$ and $\vec{b} =< 1, 0, 3 >$. Find $\vec{a} \times \vec{b}$.

Find a vector orthogonal to both $\vec{a}$ and $\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 4 \\ 1 & 0 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 4 \\ 0 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & 4 \\ 1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= \vec{i}(9) - \vec{j}(-6-4) + \vec{k}(0-3)$$

$$= 9\vec{i} + 10\vec{j} - 3\vec{k}$$

$$= \boxed{<9, 10, -3>}$$

Find the area of a parallelogram determined by $\vec{a}$ and $\vec{b}$

$$\text{area} = |\vec{a}\times\vec{b}| = \sqrt{9^2+10^2+(-3)^2} = \sqrt{81+100+9} = \sqrt{190}$$

24. Find the volume of the parallelepiped determined by vectors $\vec{a} =< 1, 0, 6 >$, $\vec{b} =< 2, 3, -8 >$, and $\vec{c} =< 8, -5, 6 >$.

$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

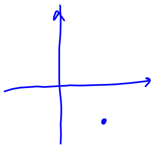
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{array}{ccc|ccc} + & + & + & - & - & - \\ 1 & 0 & 6 & 1 & 0 & 6 \\ 2 & 3 & -8 & 2 & 3 & -8 \\ 8 & -5 & 6 & 8 & -5 & 6 \end{array} = 18 - 60 - (6)(8)(3) - 40$$

$$= 18 - 100 - 144 = -226$$

$V = \boxed{226}$

25. Represent the point with Cartesian coordinates $(2\sqrt{3}, -2)$ in terms of polar coordinates.

$x = 2\sqrt{3}, \quad y = -2$ | $x = r\cos\theta$, $y = r\sin\theta$ | $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$



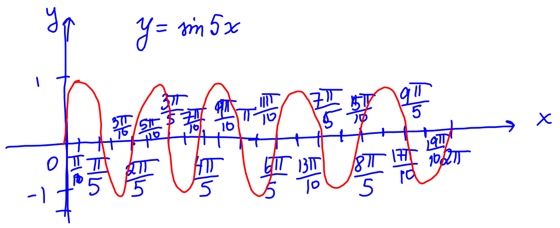
$$r^2 = x^2 + y^2 = (2\sqrt{3})^2 + (-2)^2 = 12 + 4 = 16 \Rightarrow r = 4$$

$$\tan\theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}} \qquad \frac{3\pi}{2} < \theta < 2\pi$$

$$\theta = \frac{11\pi}{6}$$

$$\left(4, -\frac{\pi}{6}\right) \text{ or } \left(4, \frac{11\pi}{6}\right) \text{ or } \left(-4, \frac{5\pi}{6}\right)$$

26. Sketch the curve $r = \sin 5\theta$.

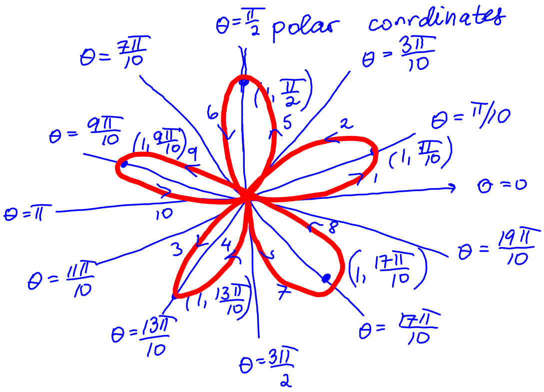


$\sin 5x = 0$

$5x = \pi n, \quad n = 0, \pm 1, \pm 2, \dots$

$x = \frac{\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \dots$

$\left(\frac{\pi}{10}, 1\right)$
$\left(\frac{3\pi}{10}, -1\right)$
$\left(\frac{\pi}{2}, 1\right)$
$\left(\frac{7\pi}{10}, -1\right)$
$\left(\frac{9\pi}{10}, 1\right)$
$\left(\frac{11\pi}{10}, -1\right)$
$\left(\frac{13\pi}{10}, 1\right)$
$\left(\frac{3\pi}{2}, -1\right)$
$\left(\frac{17\pi}{10}, 1\right)$
$\left(\frac{19\pi}{10}, -1\right)$



$\left(-1, \frac{3\pi}{10}\right) = \left(1, \frac{13\pi}{10}\right)$

$\left(-1, \frac{7\pi}{10}\right) = \left(1, \frac{17\pi}{10}\right)$

$\left(-1, \frac{11\pi}{10}\right) = \left(1, \frac{\pi}{10}\right)$