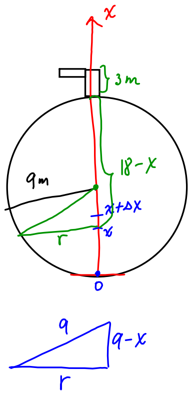
7. A tank in a shape of a sphere of radius 9 m is half full of water. Find the work W required to pump the water out of the spout, if the height of the spout is 3 m.



$r^2 = 9^2 - (9-x)^2$
$= 81 - (81 - 18x + x^2)$
$\boxed{r^2 = 18x - x^2}$

$0 \le x \le 9$ (half full of water)

slice of water between x and $x + \Delta x$

[weight of the slice] = [Volume] $(10^3)(9.81)$

slice is a cylinder of height Δx and of radius r

[Volume] $= \pi r^2 \Delta x = \pi(18x - x^2)\Delta x$

[distance travelled] $= 3 + (18 - x) = 21 - x$

[work done by pumping up the slice]

$$= \underbrace{\pi(18x - x^2)\Delta x\,(10^3)(9.81)}_{\text{weight}}\ \underbrace{(21 - x)}_{\text{distance}}$$

$$W = \pi(10^3)(9.81)\int_0^9 (18x - x^2)(21 - x)\,dx$$

(j) $\int \sin 3x \cos x \, dx$

$\sin \alpha \cos \beta = \frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$

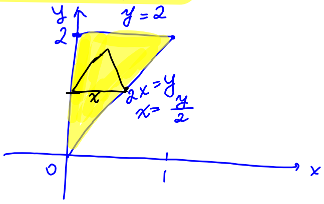
$= \int \frac{1}{2}\left[\sin(3x-x)+\sin(3x+x)\right]dx = \frac{1}{2}\int(\sin 2x+\sin 4x)\,dx$

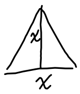
$= \boxed{\frac{1}{2}\left(-\frac{1}{2}\cos 2x-\frac{1}{4}\cos 4x\right)+C}$

6. Find the volume of the solid whose base is the triangular region with endpoints $(0,0)$, $(0,2)$, and $(1,2)$, and where cross sections perpendicular to the y-axis are isosceles triangles with height equal to base.

integrate for y. $0 \le y \le 2$

(a) $\frac{1}{24}$
(b) $\frac{2}{3}$
(c) $\frac{4}{3}$
(d) $\frac{7}{6}$
(e) $\frac{1}{3}$





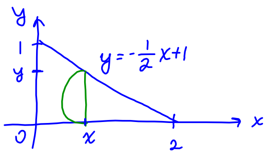
A(moving cross section) $= \frac{1}{2}x^2$

Express x in terms of y: $x = \frac{y}{2}$ (from the equation of the line through $(0,0)$ and $(1,2)$)

$A(y) = \frac{1}{2}\left(\frac{y}{2}\right)^2 = \frac{y^2}{8}$

$V = \int_0^2 A(y)\,dy = \int_0^2 \frac{y^2}{8}\,dy = \frac{y^3}{24}\Big]_0^2 = \frac{8}{24} = \frac{1}{3}$

4. The base of solid S is the triangular region with vertices (0,0), (2,0), and (0,1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.



$0 \le x \le 2$

$V = \int_0^2 [\text{area of a moving cross section}]\,dx$

y is the diameter of the cross section through x.

$[\text{area}] = \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 = \frac{\pi}{8}y^2$

equation of the line through (2,0) and (0,1).

slope $= -\frac{1}{2}$

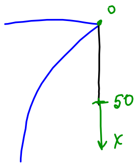
$y = -\frac{1}{2}(x-2)$

$y = -\frac{1}{2}x+1$

$A = \frac{\pi}{8}\left(-\frac{1}{2}x+1\right)^2$

$V = \frac{\pi}{8}\int_0^2 \left(-\frac{1}{2}x+1\right)^2\,dx$

5. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft hight. How much work is done in pulling the half rope to the top of the building?



$$W = \int_{25}^{50} 0.5\,x\,dx = 0.5\,\frac{x^2}{2}\Big]_{25}^{50} = \dots$$

$\int x \tan^{-1} x\,dx$ $\left| \begin{array}{ll} u = \tan^{-1}x & u' = \frac{1}{1+x^2} \\ v' = x & v = \frac{x^2}{2} \end{array} \right|$ $\int u v'\,dx = uv - \int u'v\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2}\,\frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left[\frac{\cancel{x^2+1}}{\cancel{1+x^2}} - \frac{1}{1+x^2}\right]dx$ 1

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx = \boxed{\frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right) + C}$

- $\int \sin^n x \cos^m x \, dx$
  1) if m is odd $\Rightarrow$ $u = \sin x$ $\qquad \sin^2 x + \cos^2 x = 1$
     if n is odd $\Rightarrow$ $u = \cos x$
  2) if both m and n are even, then
     $\sin^2 x = \frac{1 - \cos 2x}{2}$
     $\cos^2 x = \frac{1 + \cos 2x}{2}$

- $\int \tan^n x \sec^m x \, dx$
  1) m is even, then $u = \tan x$ $(du = \sec^2 x \, dx)$
  2) n is odd, then $u = \sec x$ $(du = \sec x \tan x \, dx)$

  $\sec^2 x = \tan^2 x + 1$
  $\tan^2 x = \sec^2 x - 1$

- $\int \sin(ax) \sin(bx) dx$
  $\int \cos(ax) \cos(bx) dx$
  $\int \sin(ax) \cos(bx) dx$

identities

$\sin\alpha \sin\beta = \frac{1}{2}\left(-\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$

$\int \cos ax \, dx = \frac{1}{a} \sin ax + C$

$\int \sin bx \, dx = -\frac{1}{b} \cos bx + C$

$\int e^{cx} dx = \frac{1}{c} e^{cx} + C$

8. The base of a solid is the region bounded by the curve $y = 4 - x^2$ and the $x-$axis. Cross sections perpendicular to the $y-$axis are squares. Find the volume of the solid.

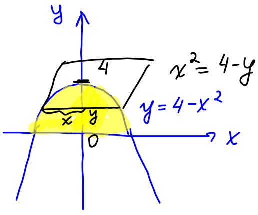
(a) $\frac{256}{3}$

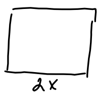
(b) 8

(c) None of these

(d) $\frac{64}{3}$

(e) 32





$A = 4x^2$
$= 4(4-y)$

Integrate for y

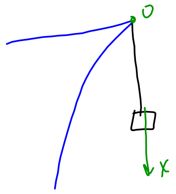
$V = \int_0^4 4(4-y)\,dy$

$= 4\int_0^4 (4-y)\,dy = 4\left(4y - \frac{y^2}{2}\right)\Big|_0^4$

$= 4\left(16 - \frac{16}{2}\right) = 4(8) = \boxed{32}$

5. A cable 40 feet long weighing 6 pounds per foot is hanging off the side of a 50 foot tall building. At the bottom of the cable is a bucket of rocks weighing 100 pounds. How much work is required to pull 10 feet of the cable to the top of the building?

   (a) 280 ft-lbs
   (b) 300 ft-lbs
   (c) 3100 ft-lbs
   (d) 2100 ft-lbs
   (e) None of these



$$W = \underbrace{W_1}_{\text{bucket}} + \overbrace{W_2}^{\text{cable}}$$

$$W_1 = \underbrace{(100)}_{\text{weight}}\overbrace{(10)}^{\text{distance}} = 1000$$

$$W_2 = \int_0^{10} 6x\,dx + \int_{10}^{40} 6(10)\,dx = 3x^2\Big]_0^{10} + 60(30) = 300 + 1800 = 2100$$

$$W = 3100$$

$$W_2 = \int_{30}^{40} 6x\,dx = 3x^2\Big]_{30}^{40} = 3(1600 - 900) = 2100$$