- THE TEST FOR DIVERGENCE: *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

- THE INTEGRAL TEST: Let $\sum a_n$ be a **positive** series. If $f$ is a continuous and decreasing function on $[a, \infty)$ such that $a_n = f(n)$ for all $n \geq a$ then $\sum a_n$ and $\int_a^{\infty} f(x)\,dx$ both converge or both diverge.

- THE COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **nonnegative** terms and $a_n \leq b_n$ for all $n$.
  1. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
  2. If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.

- LIMIT COMPARISON TEST: Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms . If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

- ALTERNATING SERIES TEST: If $b_n > 0$, $\lim_{n\to\infty} b_n = 0$ and the sequence $\{b_n\}$ is decreasing then the series $\sum(-1)^n b_n$ is convergent.

- RATIO TEST: For a series $\sum a_n$ with nonzero terms define $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.
  1. If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
  2. If $L > 1$ then the series is divergent.
  3. If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

1. Which of the following series is convergent?

   (a) $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}+1}$

$a_n = \frac{n^2}{n^{5/7}+1}$ , $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^{5/7}+1} = \infty \neq 0$

Diverges by Divergence Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$

- $-1 \le \cos n \le 1$
- $-1 \le \sin n \le 1$
- $-\frac{\pi}{2} \le \arctan n \le \frac{\pi}{2}$
- $0 \le |\cos n| \le 1$
- $0 \le |\sin n| \le 1$
- $0 \le |\arctan n| \le \frac{\pi}{2}$

$a_n = \frac{\cos^2 n}{3^n}$

Comparison Test.

$0 \le \cos^2 n \le 1$

$0 \le \frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ $\left(\begin{matrix} r = \frac{1}{3} < 1 \\ \text{geometric} \end{matrix}\right)$ – converges

By Comparison Test, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$ is $\boxed{\text{convergent}}$

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$\sum_{n=1}^{\infty} \frac{\arctan n}{n^3}$ (converges absolutely, conditionally or diverges)

$0 \le |\arctan n| \le \frac{\pi}{2}$

$0 \le \left|\frac{\arctan n}{n^3}\right| \le \frac{1}{n^3}$

since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then $\sum_{n=1}^{\infty} \left|\frac{\arctan n}{n^3}\right|$ is convergent by Comparison Test

Thus, $\sum_{n=1}^{\infty} \frac{\arctan n}{n^3}$ is $\boxed{\text{absolutely convergent}}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Integral Test.

$f(x) = \frac{1}{x(\ln x)^2} = [x(\ln x)^2]^{-1}$

Show that $f(x)$ is positive, decreasing, continuous on $[2,\infty)$

- $x>0$ and $\ln x>0$ on $[2,\infty) \Rightarrow f(x)>0$ on $[2,\infty)$
- $\frac{1}{x(\ln x)^2}$ has discontinuities at $x=0$ and $\ln x=0$, $x=1$

  $f(x)$ is continuous on $[2,\infty)$
- $f'(x) = -1\,(x(\ln x)^2)^{-2}\left((\ln x)^2 + 2x\ln x\,\frac{1}{x}\right)$

  $= -\frac{(\ln x)^2+2\ln x}{x^2(\ln x)^4} = -\frac{\ln x+2}{x^2(\ln x)^3} < 0$ on $[2,\infty)$

  $f(x)$ is decreasing on $[2,\infty)$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\int_2^t \frac{dx}{x(\ln x)^2} \quad \left|\begin{array}{l} u=\ln x, \\ du=\frac{dx}{x} \\ 2 \mapsto \ln 2 \\ t \mapsto \ln t \end{array}\right| = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u^2}$$

$$= \lim_{t\to\infty}\left(-\frac{1}{u}\right)_{\ln 2}^{\ln t} = \lim_{t\to\infty}\left(-\cancel{\frac{1}{\ln t}}^{\,0} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

converges.

By Integral Test, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ is $\boxed{\text{convergent}}$.

2. Approximate the sum of the series $\sum_{n=1}^{\infty} ne^{-n^2}$ by using the sum of first 4 terms. Estimate the error involved in this approximation.

$$\sum_{n=1}^{\infty} ne^{-n^2} \approx e^{-1} + 2e^{-4} + 3e^{-9} + 4e^{-16}$$

Error: $\int_5^{\infty} xe^{-x^2}dx \le R_4 \le \int_4^{\infty} xe^{-x^2}dx$

$$\int_a^{\infty} xe^{-x^2}dx = \left| \begin{array}{l} u=-x^2 \\ du=-2x\,dx \end{array} \right| = \lim_{t\to\infty} \left[ \int_a^t xe^{-x^2}dx \right]$$

$$= \lim_{t\to\infty} \left[ -\frac{1}{2}\int_{-a^2}^{-t^2} e^u du \right] = \lim_{t\to\infty} -\frac{1}{2}e^u \Big|_{-a^2}^{-t^2} = -\frac{1}{2}\lim_{t\to\infty} e^{-t^2} + \frac{1}{2}e^{-a^2}$$

$$= \frac{1}{2}e^{-a^2}$$

$$\boxed{\frac{1}{2}e^{-25} \le R_4 \le \frac{1}{2}e^{-16}}$$

3. Approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n^2}$ by using the sum of first 4 terms.

Estimate the error involved in this approximation.

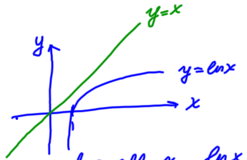
$$\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n^2} \approx e^{-1}-2e^{-4}+3e^{-9}-4e^{-16}$$

$|R_4| \leq b_5$, where $b_n = ne^{-n^2}$

$|R_4| \leq 5e^{-25}$

4. Which of the following series is absolutely convergent?

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$




for all $x$, $\ln x < x$

$\frac{1}{\ln x} > \frac{1}{x}$

$a_n = \frac{(-1)^n}{\ln n}$

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

Comparison Test.

$\frac{1}{n} < \frac{1}{\ln n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ - divergent (harmonic series)

By Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

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$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ - alternating series, $b_n = \frac{1}{\ln n}$

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$
- $\left(\frac{1}{\ln n}\right)' = -\frac{\frac{1}{n}}{(\ln n)^2} = -\frac{1}{n(\ln n)^2} < 0$ on $[2,\infty)$

$b_n$ is decreasing

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by the alternating series Test.
convergent but not absolutely convergent = $\boxed{\text{conditionally convergent}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

Ratio Test. $a_n = \frac{(-3)^n}{n!}$, $a_{n+1} = \frac{(-3)^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{\cancel{n+1}}}{(n+1)!} \frac{n!}{\cancel{(-3)^n}}\right| \underset{(n+1)!=(n+1)n!}{=} \lim_{n\to\infty} \left|\frac{-3}{n+1}\right|$$

$= 0 < 1$

$\boxed{\text{absolutely convergent by Ratio Test}}$

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

$a_n = (-1)^{n-1}\frac{1}{n}$

$\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ divergent (harmonic, $p=1$)

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ - alternating series, $b_n = \frac{1}{n}$

- $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{1}{n} = 0$
- $\left(\frac{1}{n}\right)' = -\frac{1}{n^2} < 0$ on $[1,\infty)$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent by Alternating Series Test.

$\boxed{\text{conditionally convergent}}$

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n-2}}$ .

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n}{\sqrt{n-2}}\right| = \sum_{n=1}^{\infty}\frac{n}{\sqrt{n-2}}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \infty$$

Divergent by Divergence Test.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n-2}}$$ alternating, $b_n = \frac{n}{\sqrt{n-2}}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \infty$$

$\boxed{\text{Divergent}}$

(e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{3^{3n}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{27}\right)^n$ — $\boxed{\text{absolutely convergent}}$

Ratio Test, or do $\displaystyle\sum_{n=0}^{\infty}\left(\frac{4}{27}\right)^n$ – geometric, $r=\frac{4}{27}<1$

convergent.

5. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

The radius of converges

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|, \text{ where } c_n = \frac{2^n}{\sqrt{n+3}}$$

$$R = \lim_{n\to\infty} \left| \frac{2^n}{\sqrt{n+3}} \cdot \frac{\sqrt{n+4}}{2^{n+1}} \right| = \frac{1}{2}.$$

The interval of convergence:

$$|x-3| < \frac{1}{2}$$

$$-\frac{1}{2} < x-3 < \frac{1}{2}$$

$$+\frac{5}{2} < x < \frac{7}{2}$$

End points: $x = +\frac{5}{2} \rightarrow \sum_{n=1}^{\infty} \frac{2^n\left(+\frac{5}{2}-3\right)^n}{\sqrt{n+3}}$

$$= \sum_{n=1}^{\infty} \cancel{(-1)^n} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{n+3}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} \text{ - converges but not absolutely.}$$

$x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$ diverges.

interval of convergence: $\boxed{\left[\frac{5}{2}, \frac{7}{2}\right)}$.

$\boxed{R = \frac{1}{2}}$

6. Find the power series representation for the function $f(x) = \ln(1-2x)$ centered at 0.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (+1)^n (2x)^n = \sum_{n=0}^{\infty} (+1)^n 2^n x^n$$

$$\ln(1-2x) = -2\int \frac{1}{1-2x}dx = -2\int\left(\sum_{n=0}^{\infty}(+1)^n 2^n x^n\right)dx$$

$$= -2\sum_{n=0}^{\infty}(+1)^n 2^n\left(\int x^n dx\right) = \sum_{n=0}^{\infty} -1(+2)^{n+1}\frac{x^{n+1}}{n+1} + C$$

$C$: plug $x=0$;

$\ln 1 = C \Rightarrow C = 0.$

$$\boxed{\ln(1-2x) = \sum_{n=0}^{\infty}(-1)(+2)^{n+1}\frac{x^{n+1}}{n+1}}$$

#66. Find the maclaurin series for $f(x) = \frac{x^2}{(1+9x)^3}$.

We start with $\frac{1}{1+9x} = \sum_{n=0}^{\infty} (-9x)^n = \sum_{n=0}^{\infty} (-1)^n 9^n x^n$

then, $\frac{1}{(1+9x)^2} = -\frac{1}{9}\left(\frac{1}{1+9x}\right)' = -\frac{1}{9}\left(\sum_{n=0}^{\infty} (-1)^n 9^n x^n\right)' = -\frac{1}{9}\sum_{n=1}^{\infty} (-1)^n 9^n n x^{n-1}$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} 9^{n-1} n x^{n-1}$$

$$\frac{1}{(1+9x)^3} = -\frac{1}{18}\left(\frac{1}{(1+9x)^2}\right)' = -\frac{1}{18}\left(\sum_{n=1}^{\infty} (-1)^{n+1} 9^{n-1} n x^{n-1}\right)'$$

$$= -\frac{1}{18}\sum_{n=2}^{\infty} (-1)^{n+1} 9^{n-1} n(n-1) x^{n-2}$$

$$= \frac{1}{2}\sum_{n=2}^{\infty} (-1)^n 9^{n-2} n(n-1) x^{n-2}$$

$$\frac{x^2}{(1+9x)^3} = x^2 \cdot \frac{1}{2}\sum_{n=2}^{\infty} (-1)^n 9^{n-2} n(n-1) x^{n-2}$$

$$= \boxed{\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n 9^{n-2} n(n-1) x^n}$$

7. Find the Taylor series for $f(x) = xe^{2x}$ at $x = 2$.

$$f(x) = xe^{2x}$$

$$f'(x) = e^{2x} + 2xe^{2x} = 1\cdot 2^0 e^{2x} + 2^1 x e^{2x}$$

$$f''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x}$$
$$= 4e^{2x} + 4xe^{2x} = 2\cdot 2^1 e^{2x} + 2^2 x e^{2x}$$

$$f'''(x) = 8e^{2x} + 4e^{2x} + 8xe^{2x}$$
$$= 12e^{2x} + 8xe^{2x} = 3\cdot 2^2 e^{2x} + 2^3 x e^{2x}$$

$$f^{(n)}(x) = n\cdot 2^{n-1} e^{2x} + 2^n x e^{2x}$$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(n2^{n-1} + 2^n\cdot 2)e^4}{n!}(x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n2^{n-1} + 2^{n+1})e^4}{n!}(x-2)^n$$

8. Find the Maclaurin series for $f(x) = x\sin(x^3)$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

$$x\sin x^3 = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}$$

9. Find the sum of the series

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^2}{n!}$ $\displaystyle = x^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} = x^2\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} - 1 + 1\right]$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = x^2 e^{-1}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$ $\displaystyle = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

10. Evaluate the indefinite integral as a power series $\int e^{x^2}\,dx$.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int e^{x^2} dx = \int \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \right) dx = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int x^{2n} dx \right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\, n!} + C$$

11. Approximate $f(x) = \sin x$ by a Taylor polynomial of degree 3 at $\pi/4$. ~~How accurate is this approximation if $0 \le x \le \pi/2$?~~

$$\sin x \approx f\left(\tfrac{\pi}{4}\right) + f'\left(\tfrac{\pi}{4}\right)\left(x-\tfrac{\pi}{4}\right) + f''\left(\tfrac{\pi}{4}\right)\cdot\frac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2$$
$$+ f'''\left(\tfrac{\pi}{4}\right)\frac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3$$

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{4}\right) = f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$

$$\sin x \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\tfrac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\tfrac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\tfrac{\pi}{4}\right)^3$$

12. Find radius and center of sphere given by the equation $x^2 + y^2 + z^2 = 6x + 4y + 10z$

Complete squares:

$$x^2 - 6x + y^2 - 4y + z^2 - 10z = 0$$

$$(x-3)^2 + (y-2)^2 + (z-5)^2 - 9 - 4 - 25 = 0$$

$$(x-3)^2 + (y-2)^2 + (z-5)^2 - 38 = 0$$

$$(x-3)^2 + (y-2)^2 + (z-5)^2 = 38$$

Radius $= \sqrt{38}$

Center $(3,2,5)$.