Chapter 10. **Infinite sequences and series**
Section 10.1 **Sequences**

A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, ..., a_n, ...$$

Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** or is **convergent**. Otherwise, we say the sequence **diverges** or is **divergent**

Limit Laws If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} [a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{x\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} c = c$

The Squeeze Theorem If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 1. Find the limit

(a) $\lim_{n\to\infty} (-1)^n \frac{n^2}{1+n^3}$

(b) $\lim_{n\to\infty} \frac{\cos^2 n}{2^n}$

(c) $\lim_{n\to\infty} \frac{\pi^n}{3^n}$

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Example 2. Determine whether the sequence is increasing, decreasing, or not monotonic.

(a) $a_n = \frac{1}{3n+5}$

(b) $a_n = 3 + \dfrac{(-1)^n}{n}$

(c) $a_n = \dfrac{n-2}{n+2}$

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \qquad \text{for all } n \geq 1$$

It is **bounded below** if there is a number m such that

$$a_n \geq m \qquad \text{for all } n \geq 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

Example 3. Show that the sequence defined by

$$a_1 = 2 \quad a_{n+1} = \frac{1}{3 - a_n}$$

is decreasing and bounded. Find the limit of $\{a_n\}$.