Chapter 10. **Infinite sequences and series**
Section 10.1 **Sequences**

A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, ..., a_n, ...$$

For each $n = 1, 2, 3, ...,\ a_n = f(n)$.
The sequence $a_1, a_2, ..., a_n, ...$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 1. Find the formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

1. $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$

$a_1 = 1 = \frac{1}{1} = \frac{1}{2^0}$

$a_2 = \frac{1}{2} = \frac{1}{2^1}$

$a_3 = \frac{1}{4} = \frac{1}{2^2}$

$\boxed{a_n = \frac{1}{2^{n-1}}}$

2. $\left\{\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \cdots\right\}$

$a_1 = \frac{2}{3} = \frac{2(1)}{3^1}$

$a_2 = \frac{4}{9} = \frac{2(2)}{3^2}$

$a_3 = \frac{6}{27} = \frac{2(3)}{3^3}$

$\boxed{a_n = \frac{2n}{3^n}}$

Example 2. List the first three terms of the sequence $\left\{\frac{2n+1}{4^{n-1}}\right\}$.

$a_n = \frac{2n+1}{4^{n-1}}$

$a_1 = \frac{2(1)+1}{4^{1-1}} = \frac{3}{1} = 3$

$a_2 = \frac{2(2)+1}{4^{2-1}} = \frac{5}{4}$

$a_3 = \frac{2(3)+1}{4^{3-1}} = \frac{7}{16}$

Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** or is **convergent**. Otherwise, we say the sequence **diverges** or is **divergent**

Limit Laws. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} [a_n - b_n] = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{x\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty} c = c$

The Squeeze Theorem. If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 3. Find the limit

1. $\lim_{n\to\infty} (-1)^n \frac{n^2}{1+n^3}$

$a_n = (-1)^n \frac{n^2}{1+n^3} \Rightarrow |a_n| = \frac{n^2}{1+n^3}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{1+n^3} = \lim_{n\to\infty} \frac{n^2}{n^3(\underbrace{\cancel{1/n^3}}_{0}+1)} = \lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$

By the Theorem if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$,

$\boxed{\lim_{n\to\infty} (-1)^n \frac{n^2}{1+n^3} = 0}$

2. $\lim_{n\to\infty} \frac{\cos^2 n}{2^n}$

$\frac{0}{2^n} \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n}$

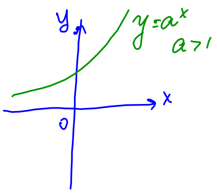
$\underbrace{0}_{a_n} \le \underbrace{\frac{\cos^2 n}{2^n}}_{b_n} \le \underbrace{\frac{1}{2^n}}_{c_n}$

2

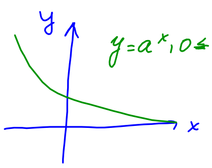
$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{2^n} = 0$, by the Squeeze Theorem, $\boxed{\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0}$

$\frac{\pi}{3} > 1$

3. $\lim_{n\to\infty} \frac{\pi^n}{3^n} = \lim \left(\frac{\pi}{3}\right)^n = \boxed{\infty}$



$\lim_{x\to\infty} a^x = \infty$



$y = a^x, 0 \le a < 1$

$\lim_{x\to\infty} a^x = 0$

Definition. A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Example 4. Determine whether the sequence is increasing, decreasing, or not monotonic.

1. $a_n = \dfrac{1}{3n+5}$

$a_n = f(n) = \frac{1}{3n+5}$

if $\{a_n\}$ is decreasing, then $f(n)$ is decreasing on $[0,\infty)$

$f'(n) < 0$ on $[0,\infty)$

$f'(n) = -\frac{1}{(3n+5)^2}(3) = -\frac{3}{(3n+5)^2} < 0$ on $[0,\infty)$

$\boxed{\text{DECREASING}}$

2. $a_n = 3 + \dfrac{(-1)^n}{n}$

$a_1 = 3 + \frac{(-1)^1}{1} = 3 - 1 = 2$

$a_2 = 3 + \frac{(-1)^2}{2} = 3 + \frac{1}{2} = \frac{7}{2}$ $a_1 < a_2$

$a_3 = 3 + \frac{(-1)^3}{3} = 3 - \frac{1}{3} = \frac{8}{3}$ $a_2 > a_3$

$\boxed{\text{NOT MONOTONIC}}$

3

3. $a_n = \dfrac{n-2}{n+2}$

$a_n = f(n) = \frac{n-2}{n+2}$

$f'(n) = \frac{1(n+2) - 1(n-2)}{(n+2)^2} = \frac{n+2-n+2}{(n+2)^2} = \frac{4}{(n+2)^2} > 0$ on $[0,\infty)$

$\boxed{\text{INCREASING}}$

Definition. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \le M \qquad \text{for all } n \ge 1$$

It is bounded below if there is a number m such that

$$a_n \ge m \qquad \text{for all } n \ge 1$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence

Monotonic Sequence Theorem. Every bounded, monotonic sequence is convergent.

Example 5. Show that the sequence defined by

$$a_1 = 2, \quad a_{n+1} = \frac{1}{3 - a_n}$$

is decreasing and bounded. Find the limit of $\{a_n\}$.

1. Decreasing ✓

$a_1 = 2$

$a_2 = \frac{1}{3-a_1} = \frac{1}{3-2} = 1$

$a_3 = \frac{1}{3-a_2} = \frac{1}{3-1} = \frac{1}{2}$

$a_1 > a_2 > a_3$

. . . .

Need to show that $a_n > a_{n+1}$ for an arbitrary $n \ge 1$.

Use math induction

Step 1. Check whether $a_1 > a_2 > a_3$ YES.

Step 2. Assume that $a_1 > a_2 > a_3 > \ldots > a_{n-1} > a_n$

Step 3. If we'll be able to show that $a_{n+1} < a_n$, then $a_1 > a_n > a_3 > \ldots > a_n > \ldots$ (the sequence'll be decreasing for all n).

We know $a_{n-1} > a_n$.

$a_{n+1} = \frac{1}{3-a_n}$ and $a_n = \frac{1}{3-a_{n-1}}$

$a_{n-1} > a_n$

$-a_{n-1} < -a_n$

$3 - a_{n-1} < 3 - a_n$

$\underbrace{\frac{1}{3-a_{n-1}}}_{a_n} > \underbrace{\frac{1}{3-a_n}}_{a_{n+1}} \Rightarrow a_n > a_{n+1}$

Conclusion: $\{a_n\}$ is $\boxed{\text{decreasing}}$

2. Bounded ✓

$a_1 = 2 > a_2 > a_3 > \ldots$

for all $n \ge 1$, $a_n \le 2$ (bounded above)

$a_{n+1} = \frac{1}{3-a_n}$, $a_n \le 2$

$0 \le a_n$ (bounded below).

bounded above and bounded below = $\boxed{\text{bounded}}$

3. Find $\lim_{n\to\infty} a_n = L$. $\cancel{0} \le L \le 2$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3 - a_{n-1}} = \frac{1}{3 - \lim_{n\to\infty} a_{n-1}}$

If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n-1} = L$

$L = \frac{1}{3-L} \Rightarrow L(3-L) = 1$

$3L - L^2 = 1$

$L^2 - 3L + 1 = 0$

$L_1 = \frac{3+\sqrt{9-1}}{2} = \frac{3+\sqrt{8}}{2} > 1$ (NOT VALID)

$L_2 = \boxed{\frac{3-\sqrt{8}}{2}}$