• Quiz Tomorrow over 9.3, 9.4, 10.1
• HW over 9.3 and 9.4 is due tomorrow night.

Section 10.2 Series

An expression of the form

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n, \quad a_n = f(n).$$

is called an **infinite series** or **series**.
Consider partial sums:

$$S_1 = a_1,$$
$$S_2 = a_1 + a_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n = a_1 + a_2 + \ldots + a_n$$

Definition. Given a series

$$\sum_{n=1}^{\infty} a_n$$

and let

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n \text{ (the } n\text{th partial sum)}$$

If the sequence

$$\{S_n\}_{n=1}^{\infty} \text{ - sequence of partial sums}$$

converges and

$$\lim_{n\to\infty} S_n = S$$

then the series is called convergent and we write

$$\sum_{n=1}^{\infty} a_n = S$$

The number S is called the sum of the series. Otherwise, the series is called divergent.

The geometric series

$$\boxed{\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1 \\ \infty, & \text{if } |r| \geq 1 \end{cases}}$$

Example 1. Find four partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n - 2}$. $\quad a_n = \frac{1}{n^2+2n-2}$

$$S_1 = a_1 = \frac{1}{1+2-2} = 1$$

$$S_2 = a_1 + a_2 = \frac{1}{1+2-2} + \frac{1}{2^2+2(2)-2}$$

$$= 1 + \frac{1}{6} = \frac{7}{6}$$

$$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3 = \frac{7}{6} + \frac{1}{3^2+2(3)-2} = \frac{7}{6} + \frac{1}{13} = \frac{7(13)+6}{6(13)}$$

$$= \frac{97}{78}$$

$$S_4 = \overbrace{a_1 + a_2 + a_3}^{S_3} + a_4 = S_3 + a_4 = \frac{97}{78} + \frac{1}{4^2+2(4)-2} = \frac{97}{78} + \frac{1}{22}$$

$S_1 = a_1,\ S_2 = a_1 + a_2 = S_1 + a_2,\ S_3 = a_1 + a_2 + a_3 = S_2 + a_3, \ldots,$

$\boxed{S_{n+1} = S_n + a_{n+1}}$ $\boxed{a_{n+1} = S_{n+1} - S_n}$

Example 2. If the nth partial sum of the series $\sum_{n=0}^{\infty} a_n$ is $S_n = \dfrac{n-1}{n+1}$, find a_n and the sum of the series S.

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}(1-\cancel{1/n}^{\,0})}{\cancel{n}(1+\cancel{1/n}^{\,0})} = \boxed{1 = S}$$

$$a_n = S_n - S_{n-1} = \underbrace{\frac{n-1}{n+1}}_{S_n} - \underbrace{\frac{(n-1)-1}{(n-1)+1}}_{S_{n-1}} = \frac{n-1}{n+1} - \frac{n-2}{n}$$

$$= \frac{n(n-1) - (n-2)(n+1)}{n(n+1)} = \frac{\cancel{n^2} - \cancel{n} - \cancel{n^2} - \cancel{n} + \cancel{2n} + 2}{n(n+1)}$$

$$\boxed{a_n = \frac{2}{n(n+1)}}$$

Example 3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, if $|r|<1$

1. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \ldots = \sum_{n=0}^{\infty} \frac{2^{n+2}}{5^n} = \sum_{n=0}^{\infty} \frac{2^2 \cdot 2^n}{5^n} = \sum_{n=0}^{\infty} 4\left(\frac{2}{5}\right)^n = \frac{4}{1-2/5} = \frac{4}{3/5} = \boxed{\frac{20}{3}}$

$a_0 = 4 = \frac{2^2}{5^0}$

$a_1 = \frac{8}{5} = \frac{2^3}{5^1}$ $\Rightarrow a_n = \frac{2^{n+2}}{5^n}$

$a_2 = \frac{16}{25} = \frac{2^4}{5^2}$

geometric, $a=4$, $r=\frac{2}{5}<1$

convergent

2. $\sum_{n=\boxed{1}}^{\infty} \frac{4^{n+1}}{5^n} = \sum_{n=1}^{\infty} \frac{4 \cdot 4^n}{5^n} = \sum_{n=\boxed{1}}^{\infty} 4\left(\frac{4}{5}\right)^n = \sum_{n=1}^{\infty} 4 \cdot \frac{4}{5}\left(\frac{4}{5}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{16}{5}\left(\frac{4}{5}\right)^{n-1}$

geometric, $a=\frac{16}{5}$, $r=\frac{4}{5}<1$ convergent

$= \frac{\frac{16}{5}}{1-\frac{4}{5}} = \frac{\frac{16}{5}}{\frac{1}{5}} = \boxed{16}$

Collapsing series.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Step 1. Partial fraction decompositing.

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$\frac{1}{n(n+2)} = \frac{A(n+2)+Bn}{n(n+2)} \Rightarrow \quad 1 = A(n+2) + Bn$$

$n=0$: $1 = 2A \Rightarrow A = \frac{1}{2}$

$n=-2$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

Step 2. Find the equation for the n-th partial term of the series S_n.

$$a_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_1 = a_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) = \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$S_2 = a_1 + a_2 = S_1 + a_2 = \underbrace{\frac{1}{2}\left(1 - \frac{1}{3}\right)}_{S_1} + \underbrace{\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)}_{a_2} = \boxed{\frac{1}{2}}\left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)$$

$$S_3 = a_1 + a_2 + a_3 = S_2 + a_3 = \underbrace{\frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)}_{S_2} + \underbrace{\frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)}_{a_3} = \boxed{\frac{1}{2}}\left(1 + \frac{1}{2} - \cancel{\frac{1}{3}} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}\right)$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4 = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{2}\left(1 + \frac{1}{2} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{5} - \frac{1}{6}\right)$$

$$= \boxed{\frac{1}{2}}\left(1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right)$$

$$\boxed{S_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)}$$

Step 3. Find $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \cancel{\frac{1}{n+1}}^{\,0} - \cancel{\frac{1}{n+2}}^{\,0}\right) = \boxed{\frac{3}{4} = S}$

Example 4. Write the number $0.\overline{307}$ as a ratio of integers.

$$0.\overline{307} = 0.\underbrace{307}\underbrace{307}\underbrace{307}\underbrace{307}\underbrace{307}\ldots$$

$$= 0.307 + 0.000307 + 0.000000307 + \ldots$$

$$= \frac{307}{1000} + \frac{307}{10^6} + \frac{307}{10^9} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{307}{(1000)^n} = \sum_{n=1}^{\infty} \frac{307}{1000}\left(\frac{1}{1000}\right)^{n-1}$$

geometric, $a = \frac{307}{1000}$, $r = \frac{1}{1000}$

$$= \frac{\frac{307}{1000}}{1 - \frac{1}{1000}} = \frac{\frac{307}{1000}}{\frac{999}{1000}} = \boxed{\frac{307}{999}}$$

- The harmonic series $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}}$
- The p-series $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent for } p > 1 \text{ and divergent for } p \leq 1}$

Theorem. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n = 0$, we can not conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

[$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$]

Test for divergence. If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5. Let $a_n = \dfrac{2n}{3n+1}$.

1. Determine whether $\{a_n\}$ is convergent. (sequence of $\{a_n\}$).

   $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$

   $\boxed{\text{THE SEQUENCE IS CONVERGENT}}$

2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

   $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$ $\boxed{\text{THE SERIES DIVERGES}}$

Theorem. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant), $\sum_{n=1}^{\infty}(a_n + b_n)$, $\sum_{n=1}^{\infty}(a_n - b_n)$, and:

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

NOTE. A finite number of terms can not affect the convergence of the series.

Example 6. Find the sum of the series $\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{6^n}$. $= \sum_{n=1}^{\infty}\left(\frac{3^n}{6^n} + \frac{(-2)^n}{6^n}\right)$

$$= \sum_{n=1}^{\infty}\left(\left(\tfrac{1}{2}\right)^n + \left(-\tfrac{1}{3}\right)^n\right) = \sum_{n=\boxed{1}}^{\infty}\left(\tfrac{1}{2}\right)^n + \sum_{n=\boxed{1}}^{\infty}\left(-\tfrac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty}\tfrac{1}{2}\left(\tfrac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty}\left(-\tfrac{1}{3}\right)\left(-\tfrac{1}{3}\right)^{n-1}$$

$$= \frac{1/2}{1-1/2} + \frac{-1/3}{1-(-1/3)} = 1 + \frac{-1/3}{4/3} = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$$