Chapter 10. **Infinite sequences and series**
Section 10.3 **The Integral and Comparison Tests; Estimating Sums**

The Integral Test Suppose f is continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

n does not have to be 1, it could be a different number. Function $f(x)$ has to be ultimately decreasing function, that is, decreasing for $x > N$.

Example 1. Determine whether the series is convergent or divergent.

(a) $\sum_{n=\boxed{2}}^{\infty} \frac{1}{n\ln n}$, $a_n = \frac{1}{n\ln n} = f(n) \Rightarrow f(x) = \frac{1}{x\ln x}$

Show that $f(x) = \frac{1}{x\ln x}$ is continuous, positive and decreasing on $[\boxed{2},\infty)$

- $f(x) = \frac{1}{x\ln x}$ has discontinuities at $x=0$ and $\ln x = 0$, $x=1$ not in $[2,\infty)$
  $f(x)$ is continuous on $[2,\infty)$
- $f(x) = \frac{1}{x\ln x}$ $x \ge 2$
  $\ln x \ge 0$ on $[1,\infty) \Rightarrow \ln x > 0$ on $[2,\infty)$
  $f(x) = \frac{1}{x\ln x} > 0$ on $[2,\infty)$
- $f'(x) = \frac{0-(x\ln x)'}{(x\ln x)^2} = \frac{-\ln x - \frac{1}{x}x}{(x\ln x)^2} = \frac{-1-\ln x}{(x\ln x)^2} < 0$ since $(x\ln x)^2 > 0$ on $[2,\infty)$
  $-1-\ln x < 0$ on $[2,\infty)$.
  $f(x)$ is decreasing on $[2,\infty)$
- So the integral $\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x}$
  $\left| \begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \\ 2 \to \ln 2 \\ t \to \ln t \end{array} \right|$

$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t\to\infty} \ln|u| \Big]_{\ln 2}^{\ln t}$

$= \lim_{t\to\infty} \left(\ln|\overset{\infty}{\cancel{\ln t}}| - \ln|\ln 2| \right) = \infty$ divergent.

- $\boxed{\sum_{n=2}^{\infty} \frac{1}{n\ln n} \text{ is divergent}}$ by the Integral Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

$f(x) = \frac{1}{x^2+4}$ on $[1,\infty)$

- $f(x) > 0$ on $[1,\infty)$
- continuous on $[1,\infty)$
- $f'(x) = -\frac{1}{(x^2+4)^2}(2x) = -\frac{2x}{(x^2+4)^2} < 0$ $x>0$ on $[1,\infty)$, $(x^2+4)^2>0$ on $[1,\infty)$.

$f(x)$ is decreasing on $[1,\infty)$

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compare $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ with $\int_1^{\infty} \frac{dx}{x^2+4} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+4}$

$= \lim_{t\to\infty} \frac{1}{2} \arctan\frac{x}{2} \Big]_1^t = \frac{1}{2}\left[\lim_{t\to\infty} \cancel{\arctan\frac{t}{2}}^{\frac{\pi}{2}} - \arctan\frac{1}{2}\right]$

$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan\frac{1}{2}\right)$ - convergent.

By the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is convergent.

Example 2. Find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent. $p > 0$

$f(n) = \frac{1}{n^p}$, $f(x) = \frac{1}{x^p}$ on $[1,\infty)$

$f(x)$ is positive, continuous on $[1,\infty)$

$f'(x) = -px^{-p-1} < 0$ on $[1,\infty)$

$f(x)$ is decreasing on $[1,\infty)$

compare $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{convergent, if } p>1 \\ \text{divergent, if } 0<p\le 1 \end{cases}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent, if } p>1 \\ \text{divergent, if } 0<p\le 1 \end{cases}}$$

The Comparison Test I Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series, such that $0 < a_n \le b_n$ for all n.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent

(b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

The Comparison Test II Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and
The Limit Comparison Test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0.$$

Then either both series converge or both diverge.

Example 3. Determine whether the series is convergent or divergent.

(a) $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$

$a_n = \frac{1}{n^2-4}$

compare with $\sum_{n=3}^{\infty} \frac{1}{n^2}$, $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1 \neq 0.$

$\sum_{n=3}^{\infty} \frac{1}{n^2}$ $(p=2>1)$ convergent. By comparison Test II, $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ is $\boxed{\text{convergent.}}$

(b) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

$\frac{0}{n\sqrt{n}} \le \frac{\sin^2 n}{n\sqrt{n}} \le \frac{1}{n\sqrt{n}} \Rightarrow 0 \le \frac{\sin^2 n}{n\sqrt{n}} \le \frac{1}{n\sqrt{n}}$

compare $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p = 3/2 > 1)$-convergent

By comparison Test I, $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$ is $\boxed{\text{convergent.}}$

(c) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

compare with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $(p = 1/2 < 1)$ – divergent

$a_n = \frac{1}{\sqrt{n}-1}$, $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}-1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}-1} = 1 \neq 0.$$

By the Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ $\boxed{\text{divergent}}$

Estimating the sum of a series

Suppose we've been able to show that a series $\sum_{n=1}^{\infty} a_n$ converges by Integral Test. We want to find an approximation to the sum S of the series. We can approximate S by partial sums S_n. How good is such an approximation?

We need to estimate the size of the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + ...$$

R_n is the error made when the partial sum S_n is used to approximate S.

$$R_n = a_{n+1} + a_{n+2} + ... \leq \int_n^{\infty} f(x)dx$$

here $f(n) = a_n$. Similarly,

$$R_n = a_{n+1} + a_{n+2} + ... \geq \int_{n+1}^{\infty} f(x)dx$$

$S = \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + \dots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + \dots}_{R_n} \Rightarrow S = S_n + R_n$

Remainder estimate for the integral test If $\sum_{n=1}^{\infty} a_n$ converges by the Integral Test, $\sum_{n=1}^{\infty} a_n = s$, $\sum_{k=1}^{n} a_k = s_n$, and $R_n = s - s_n$, then

$$\boxed{\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx}$$

or

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx}$$

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Example 4. (a) Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ by using the sum of first $\boxed{5}$ terms. Estimate the error involved in this approximation.

$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx a_1 + a_2 + a_3 + a_4 + a_5$

$= \frac{1}{1} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4}$

$= 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625}$

$S \approx S_5.$

Error: $\int_{5+1}^{\infty} \frac{dx}{x^4} \leq R_5 \leq \int_{5}^{\infty} \frac{dx}{x^4}$ or $\int_{6}^{\infty} \frac{dx}{x^4} \leq R_5 \leq \int_{5}^{\infty} \frac{dx}{x^4}$

$\int_a^{\infty} \frac{dx}{x^4} = \lim_{t\to\infty} \int_a^t \frac{dx}{x^4} = \lim_{t\to\infty} \frac{x^{-4+1}}{-4+1}\Big]_a^t$

$= \lim_{t\to\infty} \frac{x^{-3}}{-3}\Big]_a^t = \lim_{t\to\infty} \left(-\frac{1}{3x^3}\right)_a^t$

$= \lim_{t\to\infty} \left(\cancel{-\frac{1}{3t^3}}^{\,0}\right) + \frac{1}{3a^3} = \frac{1}{3a^3}$

$\boxed{\int_a^{\infty} \frac{dx}{x^4} = \frac{1}{3a^3}}$

$\frac{1}{3(6)^3} \leq R_5 \leq \frac{1}{3(5)^3}$

$\frac{1}{3(216)} \leq R_5 \leq \frac{1}{3(125)}$

$\boxed{\frac{1}{648} \leq R_5 \leq \frac{1}{375}}$

$\boxed{0.0015 \leq R_5 \leq 0.00267}$

(b) How many terms are required to ensure that the sum is accurate to within 10^{-5}?

$$R_n \le \underbrace{\int_n^\infty \frac{dx}{x^4}}_{\frac{1}{3n^3}}$$

$$R_n \le \underbrace{\frac{1}{3n^3} < 10^{-5}}$$

$$\frac{1}{3n^3} < \frac{1}{10^5} \Rightarrow 3n^3 > 10^5$$

$$n^3 > \frac{10^5}{3}$$

$$n > \sqrt[3]{\frac{10^5}{3}}$$

$$n > 32.18$$

$$\boxed{n = 33}$$