## Section 10.4 **Other Convergence Tests**

An alternating series is a series of the form

$$b_1 - b_2 + b_3 - b_4 + ... = \sum_{n=1}^{\infty}(-1)^{n+1}b_n,$$

where $b_n > 0$ for all $n$.

**The Alternating Series Test.** If the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ satisfies

(a) $b_{n+1} \leq b_n$ for all $n$ (b) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

**Example 1.** Test the series for convergence or divergence.

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2^n}$

$b_n = \frac{n}{2^n}$

$f(x) = \frac{x}{2^x}$

$f'(x) = \frac{2^x - x2^x \ln 2}{(2^x)^2} = \frac{2^x(1-x\ln 2)}{(2^x)^2} = \frac{1-x\ln 2}{2^x} < 0$

$2^x > 0$ for all $x$.

$1 - x\ln 2 < 0$

$x\ln 2 > 1$, $x > \frac{1}{\ln 2} \approx 1.44$

$\frac{x}{2^x}$ is decreasing on $\left(\frac{1}{\ln 2}, \infty\right)$

$b_2 = \frac{2}{2^2} \leq b_1$, $b_3 < b_2, \ldots$, $b_{n+1} < b_n, \ldots$

Find $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{2^n} \overset{L'H.R}{=} \lim_{n\to\infty}\frac{1}{2^n \ln 2} = 0$

$\boxed{\text{Convergent by the Alternating Series Test}}$

(b) $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{6n-5}$

$b_n = \frac{n}{6n-5}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{6n-5} = \frac{1}{6} \neq 0$

$\boxed{\text{Divergent}}$

**Alternating series estimating theorem.** If $s = \sum_{n=1}^{\infty}(-1)^{n+1}b_n$ is the sum of alternating series that satisfies the Alternating Series Test, then

$$\boxed{|R_n| = |s - s_n| \le b_{n+1}}$$

**Example 2.** Approximate the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ to four decimal places.

$$|R_n| \le 10^{-4}$$

$$|R_n| \le b_{n+1}, \quad b_n = \frac{1}{2^n n!}$$

$$b_{n+1} = \frac{1}{2^{n+1}(n+1)!} \le 10^{-4}$$

$$2^{n+1}(n+1)! \ge 10^4$$

$n=0$: $2(1)! = 2$

$n=1$: $2^2(2)! = 8$

$n=2$: $2^3 \cdot 3! = 8(3)(2) = 48$

$n=3$: $2^4 \cdot 4! = 16(4)(3)(2) = 384$

$n=4$: $2^5 \cdot 5! = 32(5)(4)(3)(2) = 3840$

$n=5$: $2^6 \cdot 6! = 64(6)(5)(4)(3)(2) = 46080 > 10000$

need $S_5$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \approx b_0 - b_1 + b_2 - b_3 + b_4 + b_5$$

$$= \frac{1}{2^0 \cdot 0!} - \frac{1}{2 \cdot 1!} + \frac{1}{2^2 \cdot 2!} - \frac{1}{2^3 \cdot 3!} + \frac{1}{2^4 \cdot 4!} - \frac{1}{2^5 5!}$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \frac{1}{3840}$$

$\boxed{n! = n(n-1)(n-2)\cdots(2)(1)}$

factorial

$0! = 1$

$1! = 1$

$2! = (2)(1) = 2$

$3! = (3)(2)(1) = 6$

. . . . . . . . . . .

$n! = n\underbrace{(n-1)(n-2)\cdots(2)(1)}_{(n-1)!}$

$$\boxed{n! = (n-1)!\,n}$$

**Definition.** A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

**Theorem.** If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

**Example 3.** Determine whether the series is absolutely convergent.

1. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$ $\boxed{\text{ABSOLUTELY CONVERGENT}}$

Remark: $n^2 > 0$ for $n \geq 1$
thus $|n^2| = n^2$.

$\sum_{n=1}^{\infty} \left| \frac{\sin 2n}{n^2} \right|$ is convergent ?

$0 \leq \frac{|\sin 2n|}{n^2} \leq \frac{1}{n^2}$

$0 \leq \frac{|\sin 2n|}{n^2} \leq \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent, by the comparison Test I, $\sum_{n=1}^{\infty} \frac{|\sin 2n|}{n^2}$ is convergent

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ alternating series.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{2n+1} \text{ - divergent.}$$

Check the Alternating Series Test.

$b_n = \frac{1}{2n+1}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

Check if $\underbrace{b_{n+1}}_{\frac{1}{2(n+1)+1}} \le \underbrace{b_n}_{\frac{1}{2n+1}}$

$b_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$ , $b_n = \frac{1}{2n+1}$

$2n+3 > 2n+1$

$\frac{1}{2n+3} < \frac{1}{2n+1}$

$b_{n+1} < b_n$ for all $n$.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ is convergent by the Alternating Series Test.

is convergent, but not absolutely convergent = $\boxed{\text{CONDITIONALLY CONVERGENT}}$

**The Ratio Test.** Given a series $\sum_{n=1}^{\infty} a_n$. Let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

1. If $L < 1$, then the series is absolutely convergent
2. If $L > 1$, then the series is divergent
3. If $L = 1$, then the test is inconclusive.

**Example 4.** Test the series for absolutely convergence, conditional convergence or divergence

1. $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$

$\boxed{\text{DIVERGENT}}$ By the Divergence Test

2. $\sum_{n=1}^{\infty} \frac{1}{n!}$

$a_n = \frac{1}{n!}$, $a_{n+1} = \frac{1}{(n+1)!}$

Do the Ratio Test.

$\boxed{(n+1)! = (n+1)n!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

$\boxed{\text{ABSOLUTELY CONVERGENT}}$ by the Ratio Test.

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{5^n}{n^2}$

$$a_n = (-1)^{n+1}\frac{5^n}{n^2}, \quad a_{n+1} = (-1)^{n+2}\frac{5^{n+1}}{(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{5^{n+1}}{(n+1)^2}}{(-1)^{n+1}\frac{5^n}{n^2}}\right| = \lim_{n\to\infty}\left|-\frac{5^{n+1}\,n^2}{(n+1)^2\,5^n}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{\cancel{5^n}\cdot 5\,n^2}{\cancel{5^n}(n+1)^2}\right| = 5\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 5 > 1$$

$\boxed{\text{DIVERGENT}}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)5^n}{n3^{2n}} = \sum_{n=1}^{\infty}\frac{(n+1)\,5^n}{n\,9^n}$

$$a_n = \frac{(n+1)\,5^n}{n\,9^n}, \quad a_{n+1} = \frac{(n+2)\,5^{n+1}}{(n+1)\,9^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+2)\,5^{n+1}}{(n+1)\,9^{n+1}}}{\frac{(n+1)\,5^n}{n\,9^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)\,5^{n+1}\,n\,9^n}{(n+1)\,9^{n+1}\,(n+1)\,5^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5(n+2)n}{9(n+1)^2}\right| = \frac{5}{9}\lim_{n\to\infty}\frac{n(n+2)}{(n+1)^2} = \frac{5}{9} < 1$$

$\boxed{\text{ABSOLUTELY CONVERGENT}}$ by the Ratio Test.

**Example 5.** For which of the following series is the Ratio Test inconclusive?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

$a_n = \frac{1}{n^3}, \quad a_{n+1} = \frac{1}{(n+1)^3}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$$

$\boxed{\text{INCONCLUSIVE}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

$a_n = \frac{n}{2^n}, \quad a_{n+1} = \frac{n+1}{2^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)2^n}{n2^{n+1}}\right| = \frac{1}{2}\lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{2} < 1$$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{1+n^2}$

$a_n = \frac{(-1)^n\sqrt{n}}{1+n^2}, \quad a_{n+1} = \frac{(-1)^{n+1}\sqrt{n+1}}{1+(n+1)^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}\sqrt{n+1}}{1+(n+1)^2}}{\frac{(-1)^n\sqrt{n}}{1+n^2}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}\sqrt{n+1}\,(1+n^2)}{[1+(n+1)^2]\,(-1)^n\sqrt{n}}\right| = 1$$

$\boxed{\text{INCONCLUSIVE}}$