Section 10.5 **Power series**

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...$$

Constants c_n are called the **coefficients** of the series. For each fixed x, the series $\sum\limits_{n=0}^{\infty} c_n x^n$ is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...$$

whose domain is the set of all x for which the series converges.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called a **power series centered at** a or a **power series about** a.

A power series is convergent if $|x-a| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

or

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}$$

R is called the **radius of convergence**.

- If $R = 0$, then the series converges only at one point $x = a$.
- If $R = \infty$, then the series converges for all x.
- If $R \neq 0$ and $R < \infty$, then the series converges if $a - R < x < a + R$. Also, test the series for convergence at $x = a - R$ and $x = a + R$.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series is convergent.

Example. Find the radius of convergence and interval of convergence for each of the following series

1. $\sum_{n=0}^{\infty} x^n$

$a_n = x^n$, $a_{n+1} = x^{n+1}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} |x| = |x| < 1$

$|x| < \boxed{1}$ ← radius of convergence. $\boxed{R=1}$

$-1 < x < 1$

$x=1$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=1}^{\infty} 1$ - diverges by the Divergence Test.

$\lim_{n\to\infty} 1 = 1 \neq 0$

$x=-1$: $\sum_{n=0}^{\infty} (-1)^n$ alternating series, $b_n = 1$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 = 1 \neq 0$

Diverges by the alternating series Test.

$\boxed{-1<x<1 \text{ - interval of convergence } (-1,1)}$

2. $\sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$

$$a_n = \frac{x^n}{(n+2)!}, \quad a_{n+1} = \frac{x^{n+1}}{((n+1)+2)!} = \frac{x^{n+1}}{(n+3)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+3)!}}{\frac{x^n}{(n+2)!}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}}{(n+3)!} \frac{(n+2)!}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x\,(n+2)!}{(n+3)!} \right|$$

$$\boxed{(n+3)! = (n+3)\underbrace{(n+2)(n+1)\cdots 2\cdot 1}_{(n+2)!} \qquad (n+3)! = (n+3)(n+2)!}$$

$$= \lim_{n\to\infty} \left| \frac{x\,\cancel{(n+2)!}}{(n+3)\cancel{(n+2)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+3} \right| = 0 < 1 \text{ for all } x.$$

$$\boxed{\begin{array}{ll} R = \infty & \text{radius of convergence.} \\ (-\infty, \infty) & \text{interval of convergence} \end{array}}$$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{\sqrt{n+3}}$

$$a_n = \frac{n!\,(x+1)^n}{\sqrt{n+3}}, \quad a_{n+1} = \frac{(n+1)!\,(x+1)^{n+1}}{\sqrt{n+4}}$$

$$\boxed{(n+1)! = (n+1)\,n!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,(x+1)^{n+1}}{\sqrt{n+4}}}{\frac{n!\,(x+1)^n}{\sqrt{n+3}}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,\cancel{(x+1)}^{n+1}\sqrt{n+3}}{n!\,\cancel{(x+1)^n}\sqrt{n+4}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\,\cancel{n!}\,(x+1)\sqrt{n+3}}{\cancel{n!}\sqrt{n+4}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\sqrt{n+3}\,(x+1)}{\sqrt{n+4}} \right| = \infty$$

$$\boxed{\begin{array}{ll} R = 0 & \text{radius of convergence} \\ x = -1 & \text{interval of convergence} \end{array}}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

$$a_n = \frac{(-1)^n x^n}{\sqrt[3]{n}}, \quad a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}}}{\frac{(-1)^n x^n}{\sqrt[3]{n}}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1} \sqrt[3]{n}}{(-1)^n x^n \sqrt[3]{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-x\sqrt[3]{n}}{\sqrt[3]{n+1}} \right| = |x| \underbrace{\lim_{n\to\infty} \left| \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \right|}_{1} = |x| < 1$$

$$|x| < 1 \Rightarrow \boxed{R = 1}$$

$$-1 < x < 1$$

$x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{\sqrt[3]{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ alternating series for $b_n = \frac{1}{\sqrt[3]{n}}$

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$
- show that $f(x) = \frac{1}{\sqrt[3]{x}}$ is decreasing.

$$f'(x) = -\frac{1}{3} x^{-\frac{1}{3}-1} = -\frac{1}{3} x^{-4/3} < 0 \quad \text{for} \quad x > 1$$

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is convergent by the Alternating Series Test.

$x = -1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt[3]{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n}}$ $\quad p = \frac{1}{3} < 1$ — divergent.

$\boxed{\text{Interval of convergence: } (-1, 1]}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n}$

$$a_n = \frac{(x-4)^n}{n\,5^n}, \quad a_{n+1} = \frac{(x-4)^{n+1}}{(n+1)\,5^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-4)^{n+1}}{(n+1)5^{n+1}}}{\frac{(x-4)^n}{n5^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}\, n\,5^n}{(n+1)(x-4)^n\,5^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(x-4)n}{5(n+1)}\right|$$

$$= \frac{|x-4|}{5} < 1$$

$$|x-4| < 5 \qquad \boxed{R=5}$$

$$-5 < x-4 < 5$$

$$-1 < x < 9$$

Test the end points.

$x=9$: $\displaystyle\sum_{n=1}^{\infty}\frac{(9-4)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{5^n}{n5^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ harmonic series divergent

$x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1-4)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n 5^n}{n\,5^n}$ $(-1)^n 5^n = (-5)^n$

$\displaystyle = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ alternating series for $b_n = \frac{1}{n}$

- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$
- $b_n = \frac{1}{n}$ is decreasing.

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is convergent.

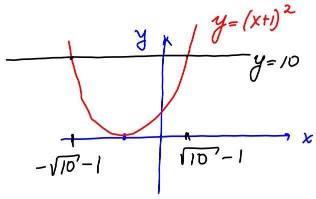
$\boxed{[-1, 9) \text{ interval of convergence}}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2(x+1)^{2n}}{10^n}$

$a_n = \dfrac{n^2(x+1)^{2n}}{10^n}$, $a_{n+1} = \dfrac{(n+1)^2(x+1)^{2(n+1)}}{10^{n+1}} = \dfrac{(n+1)^2(x+1)^{2n+2}}{10^{n+1}}$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+1)^{\cancel{2n+}2}\,\cancel{10^n}}{10^{\cancel{n+}1}\,n^2\cancel{(x+1)^{2n}}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+1)^2}{10n^2}\right|$

$= \dfrac{(x+1)^2}{10} < 1$



$(x+1)^2 < 10$

$x+1 < \sqrt{10}$ $\boxed{R=\sqrt{10}}$

$-\sqrt{10}-1 < x < \sqrt{10}-1$

Test the end points:

$n = -\sqrt{10}-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{n^2(-\sqrt{10}-1+1)^{2n}}{10^n} = \sum_{n=0}^{\infty}\frac{n^2(-\sqrt{10})^{2n}}{10^n} = \sum_{n=0}^{\infty}\frac{n^2\,10^n}{10^n}$

$\displaystyle= \sum_{n=0}^{\infty} n^2$ divergent (Divergence Test)

$\displaystyle\lim_{n\to\infty} n^2 = \infty \neq 0$

$n = \sqrt{10}-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{n^2(\sqrt{10}-1+1)^{2n}}{10^n} = \sum_{n=0}^{\infty}\frac{n^2(\sqrt{10})^{2n}}{10^n} = \sum_{n=0}^{\infty}\frac{n^2\,10^n}{10^n}$

$\displaystyle= \sum_{n=0}^{\infty} n^2$ divergent

$\boxed{(-\sqrt{10}-1,\ \sqrt{10}-1) \text{ interval of convergence}}$