Section 10.6. Representations of functions as a power series

Geometric series. geometric, $a=1$, $r=x$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1. \Rightarrow \boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$$

Example 1. Find a power series representation for the function and determine the interval of convergence.

$$\boxed{\frac{1}{1-☺} = \sum_{n=0}^{\infty} (☺)^n}$$ interval of converg. $|☺|<1$

1. $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

interval of convergence $|x|<1$.

2. $\frac{1}{4-x^2} = \frac{1}{4(1-\frac{x^2}{4})} = \frac{1}{4}\cdot\frac{1}{1-\frac{x^2}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\frac{x^{2n}}{4^n}$

$a_n = \left(\frac{x^2}{4}\right)^n$, $a_{n+1} = \left(\frac{x^2}{4}\right)^{n+1}$

$\lim_{n\to\infty}\left|\frac{(\frac{x^2}{4})^{n+1}}{(\frac{x^2}{4})^n}\right| = \lim_{n\to\infty}\left|\frac{x^2}{4}\right| = \frac{x^2}{4} < 1$

$x^2 < 4 \Rightarrow \boxed{-2<x<2}$ interval of convergence

3. $\frac{2}{1+4x^2} = 2\cdot\frac{1}{1-(-4x^2)} = 2\sum_{n=0}^{\infty}(-4x^2)^n = 2\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}$

$a_n = (-4x^2)^n$, $a_{n+1} = (-4x^2)^{n+1}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-4x^2)^{n+1}}{(-4x^2)^n}\right| = \lim_{n\to\infty}|-4x^2| = 4x^2 < 1$

$x^2 < \frac{1}{4}$

$\boxed{-\frac{1}{2}<x<\frac{1}{2}}$ interval of convergence

4. $\frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2} = -1 + 2\sum_{n=0}^{\infty}(x^2)^n = \boxed{-1+2\sum_{n=0}^{\infty} x^{2n}}$, $|x|<1$

improper fraction, need to separate the whole part.

$\boxed{-1}$ whole part

$$\begin{array}{r}
1-x^2 \,\big)\, \overline{1+x^2} \\
-1+x^2 \\
\hline
\boxed{2}
\end{array}$$

remainder

Term-by-term differentiation and integration. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + \dots + nc_n(x-a)^{n-1} + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \dots + c_n\frac{(x-a)^{n+1}}{n+1} + \dots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of these series are R. This does not mean that the interval of convergence remains the same.

Example 2. Find a power series representation for the function and determine the radius of convergence.

1. $\displaystyle\int\frac{dx}{(1+x)^2} = \frac{(1+x)^{-1}}{-1} + \cancel{C} = -\frac{1}{1+x} + \cancel{C}$ 0

$$\left(-\frac{1}{1+x}\right)' = -\frac{1}{1-(-x)} = -1\sum_{n=0}^{\infty}(-x)^n = -\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^{n+1}x^n$$

$$= \left(-1 + x - x^2 + x^3 - \dots + (-1)^{n+1}x^n + \dots\right)'$$

$$\frac{1}{(1+x)^2} = \left(-\frac{1}{1+x}\right)' = \left(-1 + x - x^2 + x^3 - \dots + (-1)^{n+1}x^n + \dots\right)'$$

$$= 1 - 2x + 3x^2 - \dots + (-1)^{n+1}nx^{n-1} + \dots$$

$$\boxed{\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}} \quad n \to n+1$$

Interval of convergence $|x|<1$

$\boxed{R=1}$

$$= \sum_{n=0}^{\infty}(-1)^{(n+1)+1}(n+1)x^{(n+1)-1}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+2}(n+1)x^n$$

$$4+x>0 \Rightarrow \boxed{x>-4.}\ !$$

2. $\ln(4+x)$

$$\int \frac{dx}{4+x} = \ln|4+x| + C$$

$$\frac{1}{4+x} = \frac{1}{4\left(1+\frac{x}{4}\right)} = \frac{1}{4}\,\frac{1}{1-\left(-\frac{x}{4}\right)} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{4^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{4^{n+1}}$$

$$\ln|4+x| = \int\frac{dx}{4+x} = \int\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{4^{n+1}}\right)dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\left(\int x^n\,dx\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,\frac{x^{n+1}}{n+1} + C$$

$$\ln|4+x| = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,\frac{x^{n+1}}{n+1} + C$$

Plug $x=0$: $\ln 4 = 0 + C \Rightarrow C = \ln 4$

$$\boxed{\ln|4+x| = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{4^{n+1}(n+1)} + \ln 4}$$

Interval of conv.

$\left|\frac{x}{4}\right|<1$

$|x|<4$

Radius of conv.

$\boxed{R=4}$

Example 3. Evaluate an indefinite integral $\int \tan^{-1}(x^2)\, dx$ as a power series.

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c \qquad \left|\begin{array}{l} u = x^2 \\ du = 2x \end{array}\right| \qquad \int \frac{du}{1+u^2} = \tan^{-1}(u) + c$$

$$\int \frac{2x\,dx}{1+x^4} = \tan^{-1}(x^2) + C$$

<u>step 1.</u> Expansion for $\frac{2x}{1+x^4} = 2x \cdot \frac{1}{1-(-x^4)} = 2x \sum_{n=0}^{\infty} (-x^4)^n$

$$= 2x \sum_{n=0}^{\infty} (-1)^n x^{4n} = \sum_{n=0}^{\infty} 2(-1)^n x^{4n+1}$$

<u>step 2.</u> $\tan^{-1}(x^2) = \int \frac{2x\,dx}{1+x^4} = \int \left(\sum_{n=0}^{\infty} 2(-1)^n x^{4n+1}\right) dx$

$$= \sum_{n=0}^{\infty} 2(-1)^n \left(\int x^{4n+1} dx\right)$$

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} 2(-1)^n \frac{x^{4n+2}}{4n+2} + C$$

Plug $x = 0$: $\tan^{-1}(0) = C \Rightarrow C = 0$

<u>Step 3.</u> $\int \tan^{-1}(x^2)\,dx = \int \left(\sum_{n=0}^{\infty} 2(-1)^n \frac{x^{4n+2}}{4n+2}\right) dx$

$$= \sum_{n=0}^{\infty} \frac{2(-1)^n}{4n+2} \left(\int x^{4n+2} dx\right)$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{2(-1)^n x^{4n+3}}{(4n+2)(4n+3)} + C}$$

Example 4. Use a power series to approximate the integral

$$\int_0^{1/2} \tan^{-1}(x^2)\, dx$$

to six decimal places.

$$\int \tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty} \frac{2(-1)^n\, x^{4n+3}}{(4n+2)(4n+3)} + C$$

$$\int_0^{1/2} \tan^{-1}(x^2)\,dx = \sum_{n=0}^{\infty} \frac{2(-1)^n}{(4n+2)(4n+3)} \left[x^{4n+3}\right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{2(-1)^n}{(4n+2)(4n+3)} \left(\frac{1}{2}\right)^{4n+3} = \sum_{n=0}^{\infty} \frac{\cancel{2}(-1)^n}{(4n+2)(4n+3)\, 2^{\cancel{4n+3}\,4n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)(4n+3)\, 2^{4n+2}}$$

$$\approx \frac{1}{6(2^2)} - \frac{1}{(6)(7)2^6} + \frac{1}{(10)(11)2^{10}} - \frac{1}{(14)(15)2^{14}}$$

$$\approx \left[0.041667 - 3.72\times10^{-4} + 8.878\times10^{-6}\right] - 2.9\times10^{-7} < 10^{-6}$$