Section 10.7 **Taylor and Maclaurin series**

Let f be any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ...(|x-a| < R)$$

Let us try to determine coefficients c_n, $n = 0, 1, 2, ...$

$$c_0 = f(a)$$

We can differentiate the series for f term-by-term.

$$f'(x) = c_1 + 2c_2(x-a) + ... + nc_n(x-a)^{n-1} + ...$$

$$c_1 = f'(a)$$

$$f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + ... + n(n-1)(x-a)^{n-2} + ...$$

$$c_2 = \frac{f''(a)}{2}$$

$$f'''(x) = 3 \cdot 2c_3 + ... + n(n-1)(n-2)(x-a)^{n-3} + ...$$

$$c_3 = \frac{f'''(a)}{3 \cdot 2} = \frac{f'''(a)}{3!}$$

So,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Theorem. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R,$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

the series is called the **Taylor series of the function f at a.**

Example 1. Find the Taylor series for the function $f(x) = \dfrac{1}{x}$ at $a = 1$.

If we plug 0 for a in the Taylor series, we'll get a series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

which is called the **Maclauren series.**

Suppose that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Let

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

T_n is called the **nth-degree Taylor polynomial of f at a** .

In general, $f(x)$ is the sum of its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$.

If we let $R_n(x)$ be the remainder of the series, then

$$R_n(x) = f(x) - T_n(x)$$

If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that $\lim_{n\to\infty} T_n(x) = f(x)$. For trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function f, we usually use the following fact.

Taylor's Inequality. If $|f^{(n+1)}(x)| \leq M$, then

$$|R_n| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

Important Maclaurin series and their intervals of convergence.

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (-1,1)}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (-\infty,\infty)}$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty,\infty)}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad (-\infty,\infty)}$$

$$\boxed{\begin{array}{l}(1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dfrac{m(m-1)(m-2)}{3!}x^3 + ... + \\ \dfrac{m(m-1)...(m-n+1)}{n!}x^n + ..., \quad [-1,1]\end{array}}$$

Example 2. Find the Maclaurin series for $f(x) = x^2 \cos(x^3)$.

Example 3. Find the sum of the series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{3^n}{n!}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!}$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

Example 4. Use series to evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}.$$

Example 5. Find the Maclaurin series for $\ln(1+x)$ and use it to calculate $\ln 1.1$ correct to five decimal places.

Example 6. Use series to approximate the definite integral $\int_0^{0.05} \cos(x^2)\ dx$ correct to three decimal places.