Section 10.7 **Taylor and Maclaurin series**

Let f be any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ...(|x-a| < R)$$

Let us try to determine coefficients c_n, $n = 0, 1, 2, ...$

$$\boxed{c_0 = f(a)}$$

We can differentiate the series for f term-by-term.

$$f'(x) = c_1 + 2c_2(x-a) + ... + nc_n(x-a)^{n-1} + ...$$

$$\boxed{c_1 = f'(a)}$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + ... + n(n-1)(x-a)^{n-2} + ...$$

$$\boxed{c_2 = \frac{f''(a)}{2}}$$

$$f'''(x) = 3\cdot 2c_3 + ... + n(n-1)(n-2)(x-a)^{n-3} + ...$$

$$\boxed{c_3 = \frac{f'''(a)}{3\cdot 2} = \frac{f'''(a)}{3!}}$$

So,

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}.}$$

Theorem. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R,$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

the series is called the Taylor series of the function f at a.

Example 1. (a) Find the Taylor series for the function $f(x) = \dfrac{1}{x}$ at $a = 1$.

$f(x) = x^{-1}$, $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$

$f'(x) = (-1)x^{-2}$

$f''(x) = (-1)(-2)x^{-3} = (-1)^2(1)(2)x^{-3}$

$f'''(x) = (-1)(-2)(-3)x^{-4} = (-1)^3(1)(2)(3)x^{-4}$

$= (-1)^3 \, 3! \, x^{-3-1}$

$f^{(n)}(x) = (-1)^n \, n! \, x^{-n-1}$

$f^{(n)}(1) = (-1)^n n!$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}}(x-1)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

(6). Find the Taylor series for $f(x) = xe^{2x}$ at $a = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$f(x) = xe^{2x}$$
$$f'(x) = e^{2x} + 2xe^{2x}$$
$$f''(x) = 2e^{2x} + 4xe^{2x} + 2e^{2x} = 4e^{2x} + 2^2 x e^{2x}$$
$$f'''(x) = 8e^{2x} + 8xe^{2x} + 4e^{2x} = 12e^{2x} + 2^3 x e^{2x} = 3\cdot 2^2 e^{2x} + 2^3 x e^{2x}$$
$$f^{(n)}(x) = n2^{n-1}e^{2x} + 2^n x e^{2x}$$
$$f^{(n)}(2) = n2^{n-1}e^4 + 2\cdot 2^n e^4$$
$$= 2^{n-1}e^4(n + 2^2) = 2^{n-1}e^4(n+4)$$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{2^{n-1}e^4(n+4)}{n!}(x-2)^n}$$

If we plug 0 for a in the Taylor series, we'll get a series

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

which is called the Maclauren series.

Suppose that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Let

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$ – partial sum for a Taylor series.

T_n is called the nth-degree Taylor polynomial of f at a .

In general, $f(x)$ is the sum of its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$.

If we let $R_n(x)$ be the remainder of the series, then

$$R_n(x) = f(x) - T_n(x)$$

If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that $\lim_{n\to\infty} T_n(x) = f(x)$. For trying to show that $\lim_{n\to\infty} R_n = 0$ for a specific function f, we usually use the following fact.

Taylor's Inequality. If $|f^{(n+1)}(x)| \le M$, then

$$\boxed{|R_n| \le \frac{M}{(n+1)!} |x-a|^{n+1}}$$

Important Maclaurin series and their intervals of convergence.

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (-1,1)}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (-\infty,\infty)}$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty,\infty)}$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad (-\infty,\infty)}$$

$$\boxed{\begin{array}{l}(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots + \\ \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots, \quad [-1,1]\end{array}}$$

Example 2. Find the Maclaurin series for $f(x) = x^2 \cos(x^3)$.

- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
- $\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!}$

  $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$
- $x^2 \cos(x^3) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$

  $= \sum_{n=0}^{\infty} (-1)^n \frac{x^2 x^{6n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(2n)!}}$

Example 3. Find the sum of the series

1. $\displaystyle\sum_{n=\boxed{2}}^{\infty} \frac{3^n}{n!}$

$$e^x = \sum_{n=\boxed{0}}^{\infty} \frac{x^n}{n!} = \underset{n=0}{1} + \underset{n=1}{\frac{x}{1}} + \underset{n=2}{\frac{x^2}{2!}} + \dots$$

$$= 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - 1 - x$$

$$\sum_{n=2}^{\infty} \frac{3^n}{n!} = e^3 - 1 - 3 = \boxed{e^3 - 4}$$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{6}\right)^{2n+1} \frac{1}{(2n+1)!} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{4}\right)^{2n} \frac{1}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Example 4. Use series to evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}.$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \overbrace{1}^{n=0} - \overbrace{\frac{x^2}{2!}}^{n=1} + \overbrace{\frac{x^4}{4!}}^{n=2} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{1}_{n=0} + \underbrace{\frac{x}{1}}_{n=1} + \underbrace{\frac{x^2}{2!}}_{n=2} + \underbrace{\frac{x^3}{3!}}_{n=3} + \dots$$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\cancel{1} - \left(\cancel{1} - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)}{\cancel{1}+\cancel{x} - \left(\cancel{1}+\cancel{x}+\frac{x^2}{2}+\frac{x^3}{6}+\dots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^4}{24} + \dots}{-\frac{x^2}{2} - \frac{x^3}{6} - \dots} = \lim_{x\to 0} \frac{\cancel{x^2}\left[\boxed{\frac{1}{2}} - \frac{x^2}{24} + \dots\right]}{\cancel{x^2}\left[\boxed{-\frac{1}{2}} - \frac{x}{6} - \dots\right]}$$

$$= \frac{\frac{1}{2}}{-\frac{1}{2}} = \boxed{-1}$$

Example 5. Find the Maclaurin series for $\ln(1+x)$ and use it to calculate $\ln 1.1$ correct to five decimal places.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \implies \boxed{\ln|1-x| = \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}}, \quad |x|<1$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n\right) dx = \sum_{n=0}^{\infty} (-1)^n \left(\int x^n\,dx\right)$$

$$\ln(1+x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Plug $x=0$: $\ln 1 = C \implies C = 0$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

$$\ln(1.1) = \ln(1+.1) = \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{n+1}}{n+1}$$

$$\approx \underbrace{0.1}_{n=0} - \underbrace{\frac{(0.1)^2}{2}}_{n=1} + \underbrace{\frac{(0.1)^3}{3}}_{n=2} - \underbrace{\frac{(0.1)^4}{4}}_{n=3} + \underbrace{\frac{(0.1)^5}{5}}_{n=4} - \cancel{\frac{(0.1)^6}{6}} < 10^{-5}$$

Example 6. Use series to approximate the definite integral $\int_0^{0.05} \cos(x^2)\,dx$ correct to three decimal places.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$\int \cos(x^2)\,dx = \int_0^{0.05} \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\int_0^{0.05} x^{4n}\,dx \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{4n+1}}{4n+1} \Big]_0^{0.05}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)} (0.05)^{4n+1}$$

$$\approx \overbrace{\frac{1}{0!}(0.05)}^{n=0} - \overbrace{\frac{1}{2!(5)}(0.05)^5}^{n=1}$$

$$= \boxed{0.05} - \frac{(0.05)^5}{10} < 10^{-3}$$