Section 10.9 **Applications of Taylor polynomials**

Suppose that

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Consider

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{6}(x-a)^3 + \cdots$$

is the nth-degree Taylor polynomial of f at a. = the n-th partial sum of the Taylor series.

Example 1. Approximate $f(x)=\sqrt{x}$ by a Taylor polynomial of degree 3 at $a=1$.

$$\sqrt{x}\approx f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{6}(x-1)^3$$

$f(x)=\sqrt{x}$	$f(1)=1$
$f'(x)=\frac{1}{2}x^{-1/2}$	$f'(1)=1/2$
$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$	$f''(1)=-\frac{1}{4}$
$f'''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$	$f'''(1)=\frac{3}{8}$

$$\sqrt{x}\approx \underbrace{1}_{f(1)}+\underbrace{\tfrac{1}{2}}_{f'(1)}(x-1)-\underbrace{\tfrac{1}{4\cdot 2}}_{\frac{f''(1)}{2}}(x-1)^2+\underbrace{\tfrac{3}{8\cdot 6}}_{\frac{f'''(1)}{6}}(x-1)^3$$

$$\boxed{\sqrt{x}\approx 1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3}$$

Example 2. Approximate $f(x)=\sin x$ by a Taylor polynomial of degree 4 at $a=\frac{\pi}{6}$.

$$\sin x\approx f\left(\tfrac{\pi}{6}\right)+f'\left(\tfrac{\pi}{6}\right)\left(x-\tfrac{\pi}{6}\right)+\frac{f''\left(\frac{\pi}{6}\right)}{2}\left(x-\tfrac{\pi}{6}\right)^2+\frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\tfrac{\pi}{6}\right)^3+\frac{f^{IV}\left(\frac{\pi}{6}\right)}{4!}\left(x-\tfrac{\pi}{6}\right)^4$$

$f(x)=\sin x$	$f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$
$f'(x)=\cos x$	$f'\left(\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$
$f''(x)=-\sin x$	$f''\left(\frac{\pi}{6}\right)=-1/2$
$f'''(x)=-\cos x$	$f'''\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$
$f^{IV}(x)=\sin x$	$f^{IV}\left(\frac{\pi}{6}\right)=\frac{1}{2}$

$$\sin x\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{6}\right)-\frac{1}{2\cdot 2}\left(x-\tfrac{\pi}{6}\right)^2-\frac{\sqrt{3}}{2\cdot 6}\left(x-\tfrac{\pi}{6}\right)^3+\frac{1}{2\cdot(24)}\left(x-\tfrac{\pi}{6}\right)^4$$

$$\boxed{\sin x\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{6}\right)-\frac{1}{4}\left(x-\tfrac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\tfrac{\pi}{6}\right)^3+\frac{1}{48}\left(x-\tfrac{\pi}{6}\right)^4}$$