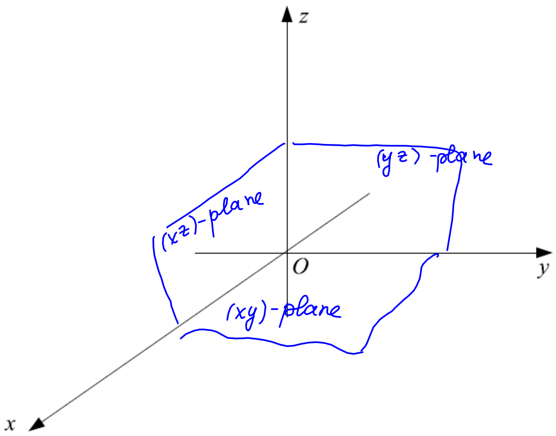
Section 11.1 **Three-dimensional coordinate system**

To locate a point in space three numbers are required. We represent any point in space by an ordered triple $(a, b, c)$.

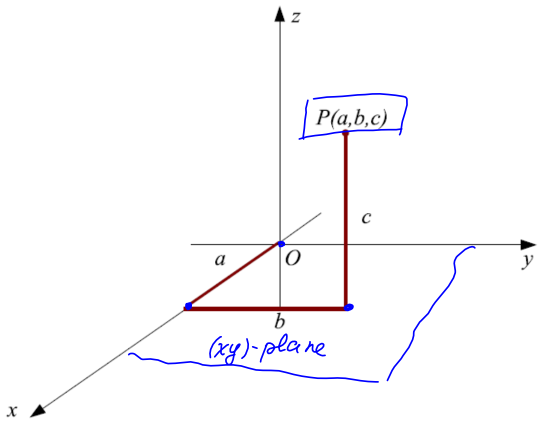
In order to represent points in space, we first choose a fixed point $O$ (the origin) and tree directed lines through $O$ that are perpendicular to each other, called the **coordinate axes** and labeled the $x$-axis, $y$-axis, and $z$-axis. Usually we think of the $x$ and $y$-axes as being horizontal and $z$-axis as being vertical.

The direction of $z$-axis is determined by the **right-hand rule**: if your index finger points in the positive direction of the $x$-axis, middle finger points in the positive direction of the $y$-axis, then your thumb points in the positive direction of the $z$-axis.



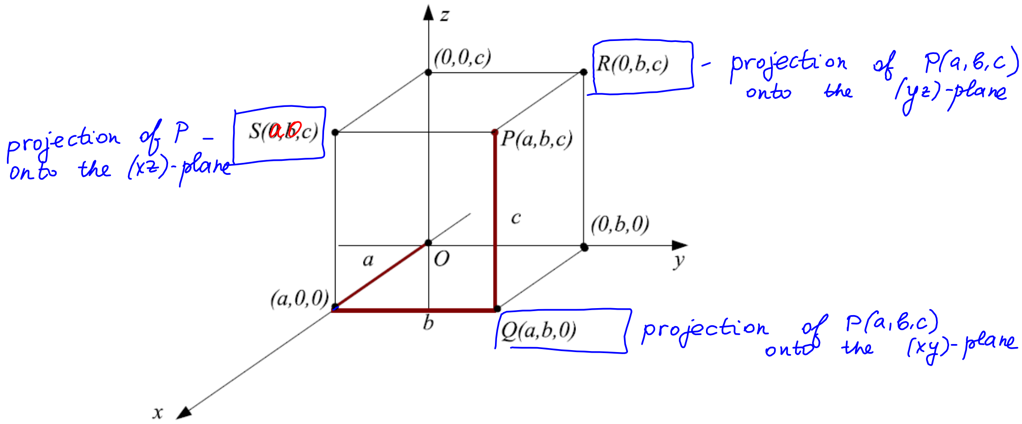
The three coordinate axes determine the three **coordinate planes**. The $xy$-plane contains the $x$- and $y$-axes and its equation is $z = 0$, the $xz$-plane contains the $x$- and $z$-axes and its equation is $y = 0$, The $yz$-plane contains the $y$- and $z$-axes and its equation is $x = 0$. These three coordinate planes divide space into eight parts called **octants**. The **first octant** is determined by positive axes.

Take a point $P$ in space, let $a$ be directed distance from $yz$-plane to $P$, $b$ be directed distance from $xz$-plane to $P$, and $c$ be directed distance from $xy$-plane to $P$.



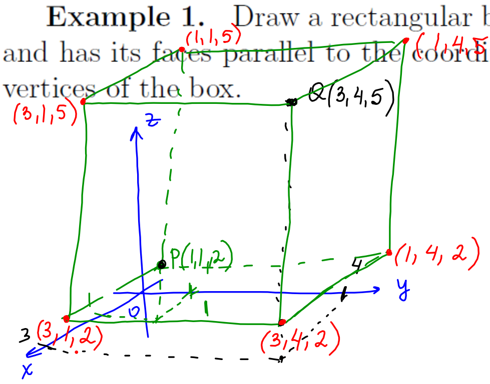
We represent the point $P$ by the ordered triple $(a, b, c)$ of real numbers, and we call $a$, $b$, and $c$ the **coordinates** of $P$.

The point $P(a, b, c)$ determine a rectangular box.




If we drop a perpendicular from $P$ to the $xy$-plane, we get a point $Q(a, b, 0)$ called the **projection** of $P$ on the $xy$-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of $P$ on the $yz$-plane and $xz$-plane, respectively.

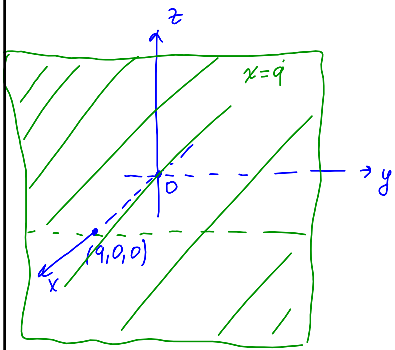
**Example 1.** Draw a rectangular box that has $P(1, 1, 2)$ and $Q(3, 4, 5)$ as opposite vertices and has its faces parallel to the coordinate planes. Then find the coordinates of the other six vertices of the box.



The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triplets of real numbers. We have given a one-to-one correspondence between points $P$ in space and ordered triplets $(a, b, c)$ in $\mathbb{R}^3$. It is called a **tree-dimensional rectangular coordinate system.**

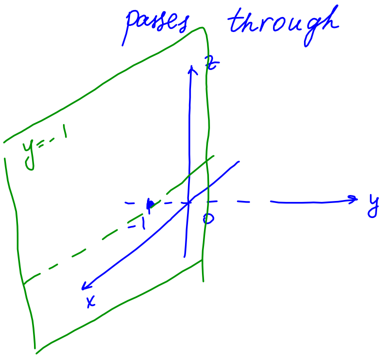
**Example 2.** What surfaces in $\mathbb{R}^3$ represented by the following equations?

1. $x = 9$ - plane
missing $y$ and $z$ $\Rightarrow$ parallel to the $(yz)$-plane
passes through $(9,0,0)$

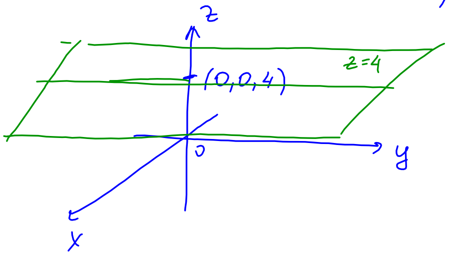



2

2. $y = -1$ - plane
$x$ and $z$ are missing $\Rightarrow$ plane parallel to the $(xz)$-plane
passes through $(0,-1,0)$

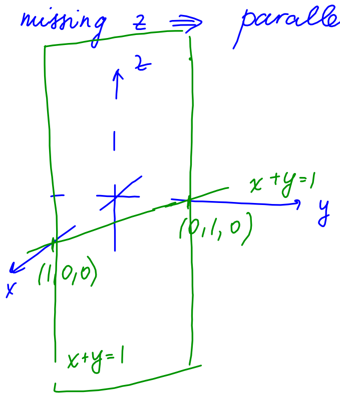


3. $z = 4$ – horizontal plane, passes through $(0,0,4)$

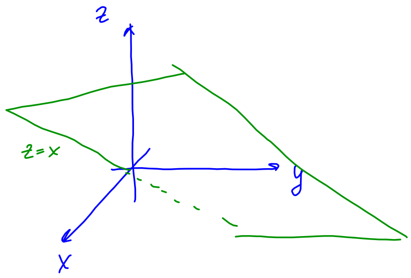



4. $x + y = 1$ powers of $x$ and $y$ $=1$ $\Rightarrow$ plane

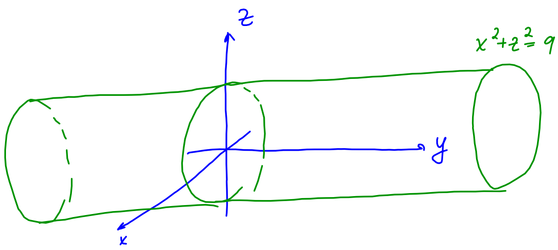
missing $z$ $\Rightarrow$ parallel to the $z$-axis



5. $z = x$ powers for both $x$ and $z = 1$ $\Rightarrow$ plane
$y$ is missing $\Rightarrow$ parallel to the $y$-axis.



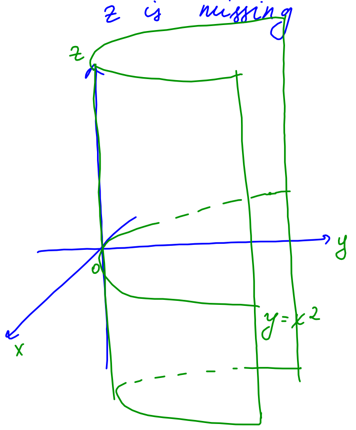

6. $x^2 + z^2 = 9$ – cylinder (circular)
$y$ is missing




3

7. $y = x^2$ - cylinder (parabolic)

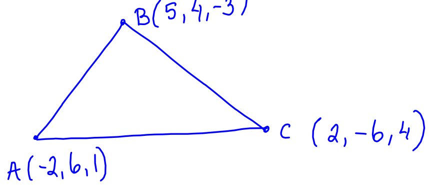
z is missing



**The distance formula in three dimensions** The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\boxed{|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

**Example 3.** Find the length of the sides of the triangle $ABC$, where $A(-2, 6, 1)$, $B(5, 4, -3)$, and $C(2, -6, 4)$.




$$|AB| = \sqrt{(5-(-2))^2 + (4-6)^2 + (-3-1)^2}$$

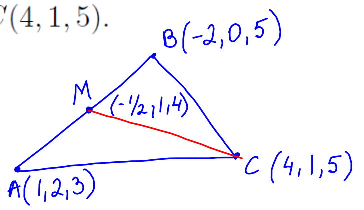
$$= \sqrt{7^2 + (-2)^2 + (-4)^2} = \sqrt{49+4+16} = \sqrt{69}$$

**Example 4.** Determine whether the points $P(1, 2, 3)$, $Q(0, 3, 7)$, and $R(3, 5, 11)$ are collinear.

The midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\boxed{P_M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)}$$

**Example 3.** Find the length of the medians of the triangle with vertices $A(1,2,3)$, $B(-2,0,5)$, and $C(4,1,5)$.




$|AM| = |MB|$

$M\left(\frac{1+(-2)}{2}, \frac{2+0}{2}, \frac{3+5}{2}\right)$

$M\left(-\frac{1}{2}, 1, 4\right)$

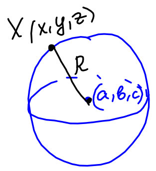
CM is the median

$|CM| = \sqrt{\left(4-\left(-\frac{1}{2}\right)\right)^2 + (1-1)^2 + (5-4)^2}$

$= \sqrt{\left(\frac{9}{2}\right)^2 + 1^2} = \sqrt{\frac{85}{4}} = \frac{\sqrt{85}}{2}$

**Definition.** A sphere is the set of all points that are equidistant from the center.

**Problem** Find an equation of a sphere of radius $R$ and center $C(a, b, c)$.




Pick an arbitrary point $X(x,y,z)$ on the sphere

$$R = |XC| = \sqrt{(a-x)^2+(b-y)^2+(c-z)^2}$$

$$(R)^2 = \left(\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}\right)^2$$

$$(a-x)^2+(b-y)^2+(c-z)^2 = R^2$$

since $(a-x)^2 = (-1)^2(x-a)^2 = (x-a)^2$

$(b-y)^2 = (-1)^2(y-b)^2 = (y-b)^2$

$(c-z)^2 = (-1)^2(z-c)^2 = (z-c)^2$

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2 = R^2}$$

equation of a sphere centered at $(a,b,c)$ of radius $R$

**Equation of a sphere** of radius $R$ and center $C(a, b, c)$ is

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2}$$

**Example 5.** Find an equation of a sphere of radius $R = 4$ centered at $C(\underbrace{-1}_{a}, \overbrace{2}^{b}, \underbrace{4}_{c})$.

$$(x+1)^2+(y-2)^2+(z-4)^2 = 4^2$$

$$\boxed{(x+1)^2+(y-2)^2+(z-4)^2 = 16}$$

**Example 6.** Find radius and center of sphere given by the equation

$$x^2 + y^2 + z^2 + x - 2y + 6z - 2 = 0$$

$$(x^2+x) + (y^2-2y) + (z^2+6z) - 2 = 0$$

complete squares:

- $x^2 + x = x^2 + 2\left(\frac{1}{2}\right)x = \underbrace{x^2 + 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2} - \left(\frac{1}{2}\right)^2$

$$= \left(x+\frac{1}{2}\right)^2 - \frac{1}{4}$$

- $y^2 - 2y = (y^2-2y+1) - 1 = (y-1)^2 - 1$
- $z^2 + 6z = z^2 + 2(3)z = (z^2 + 2(3)z + 3^2) - 3^2 = (z+3)^2 - 9$

$$(x+1/2)^2 - \frac{1}{4} + (y-1)^2 - 1 + (z+3)^2 - 9 - 2 = 0 \Rightarrow (x+1/2)^2 + (y-1)^2 + (z+3)^2 - \frac{49}{4} = 0$$

$$(x+1/2)^2 + (y-1)^2 + (z+3)^2 = \frac{49}{4}$$

$\boxed{\text{center } (-1/2, 1, -3)}$, $R = \sqrt{\frac{49}{4}} = \boxed{\frac{7}{2} = R}$

**Example 7.** Consider the points $P$ such that the distance from $P$ to $A(-1, 5, 3)$ is twice the distance from $P$ to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.

$|PA| = 2|PB|$ , $P(x, y, z)$

$|PA| = \sqrt{(-1-x)^2 + (5-y)^2 + (3-z)^2}$

$|PB| = \sqrt{(6-x)^2 + (2-y)^2 + (-2-z)^2}$

$$\left(\sqrt{(-1-x)^2 + (5-y)^2 + (3-z)^2}\right)^2 = \left(2\sqrt{(6-x)^2 + (2-y)^2 + (-2-z)^2}\right)^2$$

$$(-1-x)^2 + (5-y)^2 + (3-z)^2 = 4\left[(6-x)^2 + (2-y)^2 + (-2-z)^2\right]$$

$$1 + 2x + x^2 + 25 - 10y + y^2 + 9 - 6z + z^2 = 4\left[36 - 12x + x^2 + 4 - 4y + y^2 + 4 + 4z + z^2\right]$$

$$35 + 2x + x^2 - 10y + y^2 - 6z + z^2 = 176 - 48x + 4x^2 - 16y + 4y^2 + 16z + 4z^2$$

$$35 + 2x + x^2 - 10y + y^2 - 6z + z^2 - 176 + 48x - 4x^2 + 16y - 4y^2 - 16z - 4z^2 = 0$$

$$\frac{-3x^2 + 50x - 3y^2 + 6y - 3z^2 - 22z - 141 = 0}{-3}$$

$$\left(x^2 - \frac{50}{3}x\right) + (y^2 - 2y) + \left(z^2 + \frac{22}{3}z\right) + 47 = 0$$

$$\left[x^2 - 2\left(\frac{25}{3}\right)x + \left(\frac{25}{3}\right)^2\right] + (y^2 - 2y + 1) + \left[z^2 + 2\cdot\frac{11}{3}z + \left(\frac{11}{3}\right)^2\right] - \frac{625}{9} - 1 - \frac{121}{9} + 47 = 0$$

$$\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{332}{9}$$

$\boxed{\text{sphere centered @ } \left(\frac{25}{3}, 1, -\frac{11}{3}\right) \text{ of radius } \frac{\sqrt{332}}{3}}$