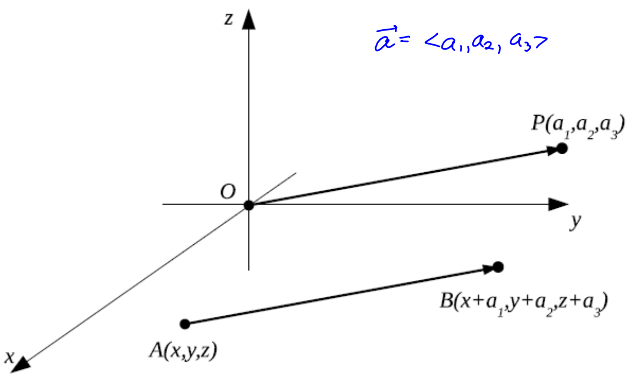
Section 11.2 **Vectors and the dot product in three dimensions**

Geometrically, a three-dimensional vector can be considered as an arrow with both a length and direction. An arrow is a directed line segment with a starting point and an ending point. Algebraically, a **tree-dimensional vector** is an ordered triple $\vec{a} =< a_1, a_2, a_3 >$ of real numbers. The numbers a_1, a_2, and a_3 are called the **components** of $\vec{a}$.

A **representation** of the vector $\vec{a} =< a_1, a_2, a_3 >$ is a directed line segment $\vec{AB}$ from any point $A(x, y, z)$ to the point $B(x + a_1, y + a_2, z + a_3)$.

A particular representation of $\vec{a} =< a_1, a_2, a_3 >$ is the directed line segment $\vec{OP}$ from the origin to the point $P(a_1, a_2, a_3)$, and $\cancel{\vec{a} =< a_1, a_2, a_3 >}$ $\vec{OP}$ is called the **position vector** of the point $P(a_1, a_2, a_3)$.

$\vec{a} = < a_1, a_2, a_3 >$

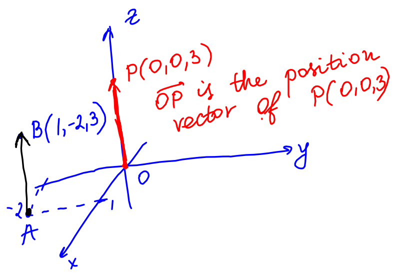


Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then $\boxed{\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >}$.

Example 1. Find a vector $\vec{a}$ with representation given by the directed line segment $\vec{AB}$, where $A(1, -2, 0)$, $B(1, -2, 3)$. Draw $\vec{AB}$ and the equivalent representation starting at the origin.

Find the components of $\vec{AB} = < 1-1, -2-(-2), 3-0 >$

$\vec{AB} = < 0, 0, 3 >$



The **magnitude (length)** $|\vec{a}|$ of $\vec{a}$ is the length of any its representation.

The length of $\vec{a}$ is $\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$

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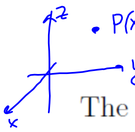
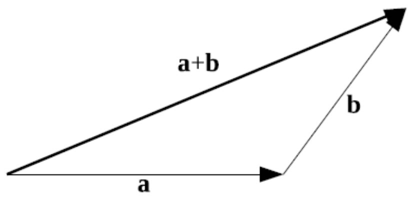


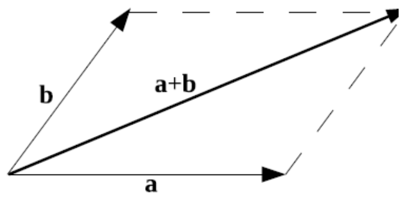
representation of $\vec{0} = <0,0,0>$

The only vector with length 0 is the **zero vector** $\vec{0} =< 0, 0, 0 >$. This vector is the only vector with no specific direction. Its representation is any point (x,y,z)

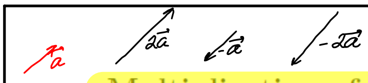
Vector addition If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then the vector $\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$



Triangle Law



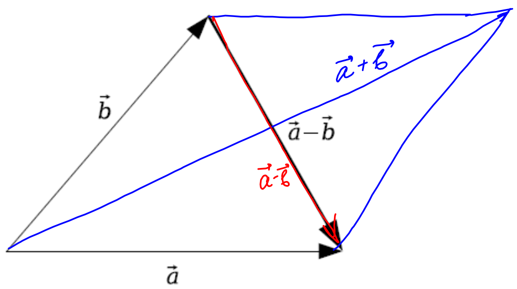
Parallelogram Law



Multiplication of a vector by a scalar If c is a scalar and $\vec{a} =< a_1, a_2, a_3 >$, then the vector $\boxed{c\vec{a} =< ca_1, ca_2, ca_3 >}$.

Two vectors $\vec{a}$ and $\vec{b}$ are called parallel if $\vec{b} = c\vec{a}$ for some scalar c. If $\vec{a} =< a_1, a_2, a_3 >$, $\vec{b} =< b_1, b_2, b_3 >$, then $\vec{a}$ and $\vec{b}$ are parallel if and only if $\boxed{\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}}$.

By the **difference** of two vectors, we mean $\boxed{\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >}$



Example 2. Find $|\vec{a}|$, $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $3\vec{b}$, $2\vec{a} - 5\vec{b}$ if $\vec{a} =< 1, -3, 2 >$, $\vec{b} =< 2, 1, -1 >$.

$\vec{a} = <1,-3,2> \Rightarrow |\vec{a}| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{1+9+4} = \sqrt{14}$

$2\vec{a} - 5\vec{b} = 2<1,-3,2> - 5<2,1,-1>$

$= <2,-6,4> - <10,5,-5>$

$= <2-10, -6-5, 4-(-5)>$

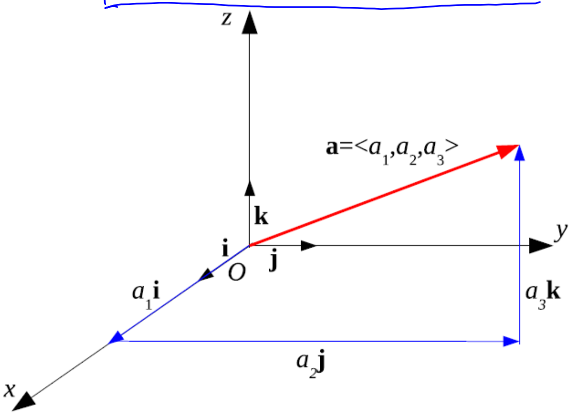
$= <-8, -11, 9>$

Properties of vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k and m are scalars, then

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
6. $(k + m)\vec{a} = k\vec{a} + m\vec{a}$
7. $(km)\vec{a} = k(m\vec{a})$
8. $1\vec{a} = \vec{a}$

Let $\vec{\imath} =< 1,0,0 >$ and $\vec{\jmath} =< 0,1,0 >$, $k =< 0,0,1 >$, $|\vec{\imath}| = |\vec{\jmath}| = |k| = 1$.

$$\boxed{\vec{a} =< a_1, a_2, a_3 >= a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}} = a_1<1,0,0> + a_2<0,1,0> + a_3<0,0,1>$$



A unit vector is a vector whose length is 1.

A vector $\boxed{\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \right\rangle}$ is a unit vector that has the same direction as $\vec{a} =< a_1, a_2, a_3 >$.

Example 3. Find the unit vector in the direction of the vector $\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$.

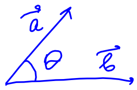
$$\vec{a} = \vec{\imath} - 2\vec{\jmath} + 2\vec{k} = <1, -2, 2>$$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

unit vector $\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}<1, -2, 2>$

$$= \boxed{< \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} >}$$

Definition. The dot or scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a}\cdot\vec{b} = 0$.



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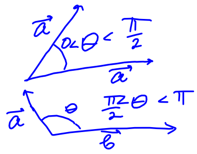
$\vec{a}\cdot\vec{b} > 0$ if and only if $0 < \theta < \pi/2$
$\vec{a}\cdot\vec{b} < 0$ if and only if $\pi/2 < \theta < \pi$



If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then $\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$.

$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$$

Example 4. Given $\vec{a} =< 2, 3, -4 >$, $\vec{b} =< 1, -4, 8 >$. Find $\vec{a}\cdot\vec{b}$.

$$\vec{a}\cdot\vec{b} = \langle 2,3,-4\rangle \cdot \langle 1,-4,8\rangle$$
$$= 2(1) + 3(-4) + (-4)8$$
$$= \boxed{-42}$$

Example 5. Find the angle between vectors $\vec{a} = 6\vec{\imath} - 2\vec{\jmath} - 3\vec{k}$ and $\vec{b} = \vec{\imath} + \vec{\jmath} + \vec{k}$.

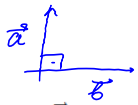
$\vec{a} = \langle 6, -2, -3\rangle$ $\quad$ $\vec{b} = \langle 1, 1, 1\rangle$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{6(1) + (-2)(1) + (-3)(1)}{\sqrt{6^2+(-2)^2+(-3)^2}\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}\sqrt{49}} = \frac{1}{7\sqrt{3}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{7\sqrt{3}}\right)}$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\pi/2$.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\boxed{\vec{a}\cdot\vec{b}=0.}$



$\cos\frac{\pi}{2}=0$

Example 6. Determine whether the vectors $\vec{a} = 3\vec{\imath} + \vec{\jmath} - \vec{k}$ and $\vec{b} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$ are orthogonal, parallel or neither.

- orthogonal $\Rightarrow$ $\vec{a}\cdot\vec{b}=0$
- parallel $\Rightarrow$ $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$

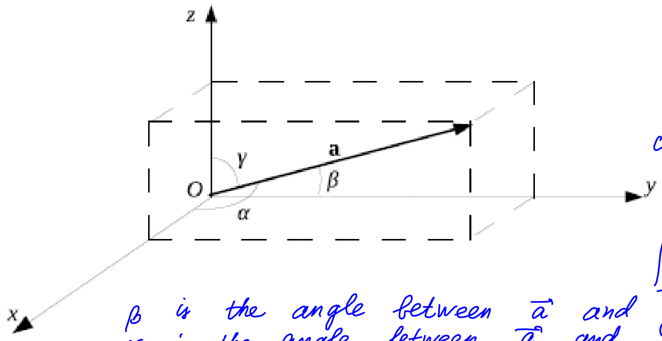
$\vec{a}=<3,1,-1>$, $\vec{b}=<1,-1,2>$

$\vec{a}\cdot\vec{b} = 3(1)+1(-1)+2(-1) = 0$

$\boxed{\text{ORTHOGONAL}}$

Example 7. Find the values of x such that the vectors $\vec{a} =< x, 1, 2 >$ and $\vec{b} =< 3, 4, x >$ are orthogonal.

Direction angles and direction cosines. The direction angles of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the direction cosines of the vector $\vec{a}$.



$\vec{a} = \langle a_1, a_2, a_3 \rangle$

α is the angle between $\vec{a}$ and $\vec{i} = \langle 1, 0, 0 \rangle$

$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1 + a_2(0) + a_3(0)}{|\vec{a}|}$

$\boxed{\cos\alpha = \frac{a_1}{|\vec{a}|}}$

β is the angle between $\vec{a}$ and $\vec{j} = \langle 0, 1, 0 \rangle$

γ is the angle between $\vec{a}$ and $\vec{k} = \langle 0, 0, 1 \rangle$

$$\boxed{\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|}}, \quad \boxed{\cos\beta = \frac{a_2}{|\vec{a}|}}, \quad \boxed{\cos\gamma = \frac{a_3}{|\vec{a}|}}.$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = 1$$

We can write

$$\vec{a} = < a_1, a_2, a_3 > = < |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma > =$$

$$\vec{a} = \quad |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} = < \cos\alpha, \cos\beta, \cos\gamma >$$

which says that the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$.

direction cosines are components of the unit vector in the direction of $\vec{a}$.

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Example 8. Find the direction cosines of the vector $\vec{a} = < -4, -1, 2 >$.

$$|\vec{a}| = \sqrt{(-4)^2 + (-1)^2 + 2^2} = \sqrt{16+1+4} = \sqrt{21}$$

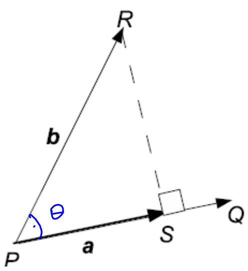
$$\frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{21}} < -4, -1, 2 > = < \underbrace{-\frac{4}{\sqrt{21}}}_{\cos\alpha}, \underbrace{-\frac{1}{\sqrt{21}}}_{\cos\beta}, \underbrace{\frac{2}{\sqrt{21}}}_{\cos\gamma} >$$

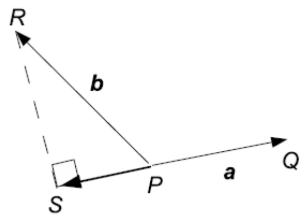
$$\boxed{\begin{aligned} \cos\alpha &= -\frac{4}{\sqrt{21}} \\ \cos\beta &= -\frac{1}{\sqrt{21}} \\ \cos\gamma &= \frac{2}{\sqrt{21}} \end{aligned}}$$

$\text{comp}_{\vec a}\vec b = |\vec{PS}| = |\vec{PR}|\cos\theta$

$= |\vec b| \cdot \dfrac{\vec a\cdot\vec b}{|\vec a|\,|\vec b|}$

$= \dfrac{\vec a\cdot\vec b}{|\vec a|}$





$\frac{\pi}{2} < \theta < \pi$

$\vec{SP} = \text{proj}_{\vec a}\vec b$

$\text{comp}_{\vec a}\vec b = -|\vec{SP}|$

$\vec{PS} = \text{proj}_{\vec a}\vec b$ is called the vector projection **of $\vec b$ onto $\vec a$.**

$0 < \theta < \frac{\pi}{2}$

$|\vec{PS}| = \text{comp}_{\vec a}\vec b$ is called the scalar projection **of $\vec b$ onto $\vec a$** or the **component of $\vec b$ along** $\vec a$.

$$\boxed{\text{comp}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|}} \quad \boxed{\text{proj}_{\vec a}\vec b = \frac{\vec a\cdot\vec b}{|\vec a|^2}\vec a = \frac{\vec a\cdot\vec b}{|\vec a|^2} < a_1, a_2, a_3 >}$$

$\text{proj}_{\vec a}\vec b = (\text{comp}_{\vec a}\vec b)\dfrac{\vec a}{|\vec a|}$

Example 9. Find the scalar and vector projections of $\vec b =< 4, 2, 0 >$ onto $\vec a =< 1, 2, 3 >$.

$\text{comp}_{\vec a}\vec b = \dfrac{\vec a\cdot\vec b}{|\vec a|} = \dfrac{4(1)+2(2)+(0)(3)}{\sqrt{1^2+2^2+3^2}} = \boxed{\dfrac{8}{\sqrt{14}}}$ - scalar projection

$\text{proj}_{\vec a}\vec b = (\text{comp}_{\vec a}\vec b)\cdot\dfrac{\vec a}{|\vec a|} = \dfrac{8}{\sqrt{14}} \cdot \dfrac{<1,2,3>}{\sqrt{14}} = \dfrac{8}{14} <1,2,3>$

$= \boxed{< \frac{4}{7}, \frac{8}{7}, \frac{12}{7} >}$ - vector projection

Example 10. A constant force with vector representation $\vec{F} = 10\vec{\imath} + 18\vec{\jmath} - 6\vec{k}$ moves an object along a straight line from the point $A(2,3,0)$ to the point $B(4,9,15)$. Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

$W = \vec{F} \cdot \vec{D}$

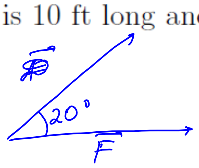
$\vec{D} = \vec{AB} = \langle 4-2, 9-3, 15-0 \rangle$

$= \langle 2, 6, 15 \rangle$

$\vec{F} = \langle 10, 18, -6 \rangle$

$W = \langle 10, 18, -6 \rangle \cdot \langle 2, 6, 15 \rangle = 10(2) + 18(6) - 6(15) = \boxed{38\ (J)}$

Example 11. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20^0 above the horizontal. How much work is done?



$|\vec{F}| = 25, \quad |\vec{D}| = 10$

$\theta = 20°$

$W = |\vec{F}||\vec{D}| \cos 20° = \vec{F} \cdot \vec{D}$

$= (25)(10)\cos(20°) \approx \boxed{234.92\ (lb\text{-}ft)}$